V. A. KRECHMAR

A PROBLEM BOOK IN ALGEBRA

MIR PUBLISHERS · MOSCOW

ИЗДАТЕЛЬСТВО МИР

В А КРЕЧМАР

ЗАДАЧНИК ПО АЛГЕБРЕ

Издательство «Наука»

V. A. KRECHMAR

A PROBLEM BOOK IN ALGEBRA

Translated from the Russian
by
Victor Shiffer

Translation editor Leonid Levant

MIR PUBLISHERS
MOSCOW

First published 1974

Second printing 1978

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is:
USSR, 129820, Moscow I-110, GSP
Pervy Rizhsky Pereulok, 2
MIR PUBLISHERS

На английском языке

Printed in the Union of Soviet Socialist Republics

CONTENTS

PROBLEMS

1. WHOLE RATIONAL EXPRESSIONS

The problems presented in this section are mainly on the identity transformations of whole rational expressions. These are the elementary operations we have to use here: addition, multiplication, division and subtraction of monomials and polynomials, as well as raising them to various powers and resolving them into factors. As regards trigonometric problems, we take as known the definition of trigonometric functions, principal relationships between these functions, all the properties connected with their periodicity, and all corollaries of the addition theorem.

Attention should be drawn only to the formulas which enable us to transform a product of trigonometric functions into a sum or a difference of these functions. Namely:

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)],$$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)],$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)].$$

1. Prove the identity

$$(a^2+b^2)(x^2+y^2) = (ax-by)^2 + (bx+ay)^2.$$

2. Show that

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2) = \\ = (ax-by-cz-dt)^2 + (bx+ay-dz+ct)^2 + \\ + (cx+dy+az-bt)^2 + (dx-cy+bz+at)^2.$$

3. Prove that from the equalities

$$ax-by-cz-dt=0, \quad bx+ay-dz+ct=0,$$

$$cx+dy+az-bt=0, \quad dx-cy+bz+at=0,$$

follows either $a = b = c = d = 0$, or $x = y = z = t = 0$.

4. Show that the following identity takes place

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 =$$
$$= (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2.$$

5. Show that the preceding identity can be generalized in the following way

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) =$$
$$= (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 + (a_1b_2 - a_2b_1)^2 +$$
$$+ (a_1b_3 - a_3b_1)^2 + \ldots + (a_{n-1}b_n - a_nb_{n-1})^2.$$

6. Let

$$n(a^2 + b^2 + c^2 + \ldots + l^2) =$$
$$= (a + b + c + \ldots + l)^2,$$

where n is the number of the quantities $a, b, c, \ldots, l$.

Prove that then

$$a = b = c = \ldots = l.$$

7. Prove that from the equalities

$$a_1^2 + a_2^2 + \ldots + a_n^2 = 1, \quad b_1^2 + b_2^2 + \ldots + b_n^2 = 1$$

follows

$$-1 \leqslant a_1b_1 + a_2b_2 + \ldots + a_nb_n \leqslant +1.$$

8. Prove that from the equality

$$(y - z)^2 + (z - x)^2 + (x - y)^2 =$$
$$= (y + z - 2x)^2 + (z + x - 2y)^2 + (x + y - 2z)^2$$

follows

$$x = y = z.$$

9. Prove the following identities

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2,$$
$$(6a^2 - 4ab + 4b^2)^3 = (3a^2 + 5ab - 5b^2)^3 +$$
$$+ (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3.$$

10. Show that

$$(p^2 - q^2)^4 + (2pq + q^2)^4 + (2pq + p^2)^4 = 2(p^2+pq+q^2)^4.$$

11. Prove the identity

$$X^2 + XY + Y^2 = Z^3$$

if

$$X = q^3 + 3pq^2 - p^3, \quad Y = -3pq(p+q),$$
$$Z = p^2 + pq + q^2.$$

12. Prove that

$$(3a+3b)^k + (2a+4b)^k + a^k + b^k = (3a+4b)^k + (a+3b)^k + (2a+b)^k$$

at $k = 1, 2, 3$.

13. 1° Show that if $x + y + z = 0$, then

$$(ix - ky)^n + (iy - kz)^n + (iz - kx)^n = (iy - kx)^n + (iz - ky)^n + (ix - kz)^n$$

at $n = 0, 1, 2, 4$.

2° Prove that

$$x^n + (x+3)^n + (x+5)^n + (x+6)^n + (x+9)^n + (x+10)^n + (x+12)^n + (x+15)^n = (x+1)^n + (x+2)^n + (x+4)^n + (x+7)^n + (x+8)^n + (x+11)^n + (x+13)^n + (x+14)^n$$

at $n = 0, 1, 2, 3$.

14. Prove the identities

1° $(a+b+c+d)^2 + (a+b-c-d)^2 + (a+c-b-d)^2 + (a+d-b-c)^2 = 4(a^2+b^2+c^2+d^2)$;

2° $(a^2-b^2+c^2-d^2)^2 + 2(ab-bc+dc+ad)^2 = (a^2+b^2+c^2+d^2)^2 - 2(ab-ad+bc+dc)^2$;

3° $(a^2-c^2+2bd)^2 + (d^2-b^2+2ac)^2 = (a^2-b^2+c^2-d^2)^2 + 2(ab-bc+dc+ad)^2$.

15. Prove the identity

$$(a+b+c)^4 + (b+c-a)^4 + (c+a-b)^4 + (a+b-c)^4 = 4(a^4+b^4+c^4) + 24(b^2c^2+c^2a^2+a^2b^2).$$

16. Let $s = a + b + c$.
Prove that

$$s(s-2b)(s-2c)+s(s-2c)(s-2a)+ \\ +s(s-2a)(s-2b)=(s-2a)(s-2b)(s-2c)+8abc.$$

17. Prove that if $a + b + c = 2s$, then

$$a(s-a)^2+b(s-b)^2+c(s-c)^2+2(s-a)\times \\ \times(s-b)(s-c)=abc.$$

18. Put

$$2s = a + b + c; \quad 2\sigma^2 = a^2 + b^2 + c^2.$$

Show that

$$(\sigma^2-a^2)(\sigma^2-b^2)+(\sigma^2-b^2)(\sigma^2-c^2)+ \\ +(\sigma^2-c^2)(\sigma^2-a^2)=4s(s-a)(s-b)(s-c).$$

19. Factor the following expression

$$(x+y+z)^3-x^3-y^3-z^3.$$

20. Factor the following expression

$$x^3+y^3+z^3-3xyz.$$

21. Simplify the expression

$$(a+b+c)^3-(a+b-c)^3-(b+c-a)^3- \\ -(c+a-b)^3.$$

22. Factor the following expression

$$(b-c)^3+(c-a)^3+(a-b)^3.$$

23. Show that if $a + b + c = 0$, then

$$a^3+b^3+c^3=3abc.$$

24. Prove that if $a + b + c = 0$, then

$$(a^2+b^2+c^2)^2=2(a^4+b^4+c^4).$$

25. Show that

$$[(a-b)^2+(b-c)^2+(c-a)^2]^2= \\ =2[(a-b)^4+(b-c)^4+(c-a)^4].$$

26. Let $a + b + c = 0$, prove that

1° $2(a^5 + b^5 + c^5) = 5abc(a^2 + b^2 + c^2)$;

2° $5(a^3 + b^3 + c^3)(a^2 + b^2 + c^2) = 6(a^5 + b^5 + c^5)$;

3° $10(a^7 + b^7 + c^7) = 7(a^2 + b^2 + c^2)(a^5 + b^5 + c^5)$.

27. Given $2n$ numbers: $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$. Put

$$b_1 + b_2 + \ldots + b_n = s_n.$$

Prove that

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = (a_1 - a_2)s_1 + (a_2 - a_3)s_2 + \\ + \ldots + (a_{n-1} - a_n)s_{n-1} + a_ns_n.$$

28. Put

$$a_1 + a_2 + \ldots + a_n = \frac{n}{2}s.$$

Prove that

$$(s - a_1)^2 + (s - a_2)^2 + \ldots + (s - a_n)^2 = \\ = a_1^2 + a_2^2 + \ldots + a_n^2.$$

29. Given a trinomial $Ax^2 + 2Bxy + Cy^2$.
Put

$$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y'.$$

Then the given trinomial becomes

$$A'x'^2 + 2B'x'y' + C'y'^2.$$

Prove that

$$B'^2 - A'C' = (B^2 - AC)(\alpha\delta - \beta\gamma)^2.$$

30. Let

$$p_i + q_i = 1 \quad (i = 1, 2, \ldots, n)$$

and

$$p = \frac{p_1 + p_2 + \ldots + p_n}{n}, \quad q = \frac{q_1 + q_2 + \ldots + q_n}{n}.$$

Prove that

$$p_1q_1 + p_2q_2 + \ldots + p_nq_n = npq - (p_1 - p)^2 - \\ - (p_2 - p)^2 - \ldots - (p_n - p)^2.$$

31. Prove that

$$\frac{1}{1}\cdot\frac{1}{2n-1}+\frac{1}{3}\cdot\frac{1}{2n-3}+\ldots+\frac{1}{2n-1}\cdot\frac{1}{1}=$$
$$=\frac{1}{n}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right).$$

32. Let $s_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

Show that

1° $s_n=n-\left(\frac{1}{2}+\frac{2}{3}+\ldots+\frac{n-1}{n}\right)$;

2° $ns_n=n+\left(\frac{n-1}{1}+\frac{n-2}{2}+\ldots+\frac{2}{n-2}+\frac{1}{n-1}\right)$.

33. Prove the identity

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+$$
$$+\frac{1}{2n}.$$

34. Prove

$$\left(1+\frac{1}{\alpha-1}\right)\left(1-\frac{1}{2\alpha-1}\right)\left(1+\frac{1}{3\alpha-1}\right)\times\ldots\times$$
$$\times\left(1+\frac{1}{(2n-1)\alpha-1}\right)\left(1-\frac{1}{2n\alpha-1}\right)=$$
$$=\frac{(n+1)\alpha}{(n+1)\alpha-1}\cdot\frac{(n+2)\alpha}{(n+2)\alpha-1}\cdots\frac{(n+n)\alpha}{(n+n)\alpha-1}.$$

35. Let $[\alpha]$ denote the whole number nearest to α which is less than or equal to it. Thus, $[\alpha]\leqslant\alpha<[\alpha]+1$. Prove that there exists the identity

$$[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right]=[nx].$$

36. Prove that

$$\cos(a+b)\cos(a-b)=\cos^2 a-\sin^2 b.$$

37. Show that

$$(\cos a+\cos b)^2+(\sin a+\sin b)^2=4\cos^2\frac{a-b}{2},$$

$$(\cos a-\cos b)^2+(\sin a-\sin b)^2=4\sin^2\frac{a-b}{2}.$$

38. Given

$$(1+\sin a)(1+\sin b)(1+\sin c)=\cos a\cos b\cos c.$$

Simplify

$$(1-\sin a)(1-\sin b)(1-\sin c).$$

39. Given

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)=$$
$$=(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma).$$

Show that one of the values of each member of this equality is

$$\sin\alpha\sin\beta\sin\gamma.$$

40. Show that

$$\cos(\alpha+\beta)\sin(\alpha-\beta)+\cos(\beta+\gamma)\sin(\beta-\gamma)+$$
$$+\cos(\gamma+\delta)\sin(\gamma-\delta)+\cos(\delta+\alpha)\sin(\delta-\alpha)=0.$$

41. Prove that

$$\sin(a+b)\sin(a-b)\sin(c+d)\sin(c-d)+$$
$$+\sin(c+b)\sin(c-b)\sin(d+a)\sin(d-a)+$$
$$+\sin(d+b)\sin(d-b)\sin(a+c)\sin(a-c)=0.$$

42. Check the identities:

1° $\cos(\beta+\gamma-\alpha)+\cos(\gamma+\alpha-\beta)+$
$+\cos(\alpha+\beta-\gamma)+\cos(\alpha+\beta+\gamma)=4\cos\alpha\cos\beta\cos\gamma;$

2° $\sin(\alpha+\beta+\gamma)+\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)-$
$-\sin(\alpha+\beta-\gamma)=4\cos\alpha\cos\beta\sin\gamma.$

43. Reduce the following expression to a form convenient for taking logarithms

$$\sin\left(A+\frac{B}{4}\right)+\sin\left(B+\frac{C}{4}\right)+\sin\left(C+\frac{A}{4}\right)+$$
$$+\cos\left(A+\frac{B}{4}\right)+\cos\left(B+\frac{C}{4}\right)+\cos\left(C+\frac{A}{4}\right)$$

if $A+B+C=\pi$.

44. Reduce the following expression to a form convenient for taking logarithms

$$\sin\frac{A}{4}+\sin\frac{B}{4}+\sin\frac{C}{4}+\cos\frac{A}{4}+\cos\frac{B}{4}+\cos\frac{C}{4}$$

if $A+B+C=\pi$.

45. Simplify the product

$$\cos a\cos 2a\cos 4a\ldots\cos 2^{n-1}a.$$

46. Show that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\left(\frac{1}{2}\right)^7.$$

47. Given $\sin B=\frac{1}{5}\sin(2A+B)$.

Prove that

$$\tan(A+B)=\frac{3}{2}\tan A.$$

48. Let A and B be acute positive angles satisfying the equalities

$$3\sin^2 A+2\sin^2 B=1,$$

$$3\sin 2A-2\sin 2B=0.$$

Prove that $A+2B=\frac{\pi}{2}$.

49. Show that the magnitude of the expression

$$\cos^2\varphi+\cos^2(a+\varphi)-2\cos a\cos\varphi\cos(a+\varphi)$$

is independent of φ.

50. Let

$$a=\cos\varphi\cos\psi+\sin\varphi\sin\psi\cos\delta,$$

$$a'=\cos\varphi\sin\psi-\sin\varphi\cos\psi\cos\delta,\quad a''=\sin\varphi\sin\delta;$$

$$b=\sin\varphi\cos\psi-\cos\varphi\sin\psi\cos\delta,$$

$$b'=\sin\varphi\sin\psi+\cos\varphi\cos\psi\cos\delta,\quad b''=-\cos\varphi\sin\delta;$$

$$c=-\sin\psi\sin\delta,\quad c'=\cos\psi\sin\delta,\quad c''=\cos\delta.$$

Prove that

$$a^2 + a'^2 + a''^2 = 1, \quad b^2 + b'^2 + b''^2 = 1,$$
$$c^2 + c'^2 + c''^2 = 1,$$
$$ab + a'b' + a''b'' = 0, \quad ac + a'c' + a''c'' = 0,$$
$$bc + b'c' + b''c'' = 0.$$

2. RATIONAL FRACTIONS

Transformations of fractional rational expressions to be considered in this section are based on standard rules of operations with algebraic fractions.

Let us draw our attention only to one point which we have to use (see Problems 15, 16, 17). If we have a first-degree binomial in x

$$Ax + B$$

and if we know that it vanishes at two different values of x (say, at $x = a$ and $x = b$), then we may state that the coefficients A and B are equal to zero. Indeed, from the equalities

$$Aa + B = 0, \quad Ab + B = 0 \qquad (*)$$

we get

$$A(a - b) = 0$$

and since $a - b \neq 0$, then $A = 0$. Substituting this value into one of the equalities (*), we find $B = 0$. Similarly, we may assert that if a second-degree trinomial in x

$$Ax^2 + Bx + C$$

vanishes at three distinct values of x (say, at $x = a$, $x = b$ and $x = c$), then $A = B = C = 0$.

Indeed, we then have

$$Aa^2 + Ba + C = 0, \quad Ab^2 + Bb + C = 0,$$
$$Ac^2 + Bc + C = 0.$$

Subtracting term by term, we have

$$A(a^2 - b^2) + B(a - b) = 0, \quad A(a^2 - c^2) + B(a - c) = 0.$$

Since $a-b\neq 0$, $a-c\neq 0$, we have

$$A(a+b)+B=0, \quad A(a+c)+B=0.$$

Hence $A=0$ (since $b-c\neq 0$), and then we find $B=0$ and $C=0$.

Analogously, we can show that if a third-degree polynomial

$$Ax^3+Bx^2+Cx+D$$

vanishes at four different values of x, then

$$A=B=C=D=0,$$

and, in general, if an nth-degree polynomial vanishes at $n+1$ different values of x, then its coefficients are equal to zero (see Sec. 6).

Finally, considered in this section are a number of problems pertaining finite continued fractions. We take as known the information on these fractions contained usually in elementary textbooks.

The principal trigonometric relations used in solving triangles are also taken here as known.

1. Prove the identity

$$p^3=\left(p\frac{p^3-2q^3}{p^3+q^3}\right)^3+\left(q\frac{2p^3-q^3}{p^3+q^3}\right)^3+q^3.$$

2. Simplify the following expression

$$\frac{1}{(p+q)^3}\left(\frac{1}{p^3}+\frac{1}{q^3}\right)+\frac{3}{(p+q)^4}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)+\frac{6}{(p+q)^5}\left(\frac{1}{p}+\frac{1}{q}\right).$$

3. Simplify

$$\frac{1}{(p+q)^3}\left(\frac{1}{p^4}-\frac{1}{q^4}\right)+\frac{2}{(p+q)^4}\left(\frac{1}{p^3}-\frac{1}{q^3}\right)+$$
$$+\frac{2}{(p+q)^5}\left(\frac{1}{p^2}-\frac{1}{q^2}\right).$$

4. Let

$$x=\frac{a-b}{a+b}, \quad y=\frac{b-c}{b+c}, \quad z=\frac{c-a}{c+a}.$$

Prove that

$$(1+x)(1+y)(1+z)=(1-x)(1-y)(1-z).$$

5. Show that from the equality

$$(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$$

follows

$$\frac{a}{c}=\frac{b}{d}.$$

6. Simplify the expression

$$\frac{ax^2+by^2+cz^2}{bc(y-z)^2+ca(z-x)^2+ab(x-y)^2}$$

if

$$ax+by+cz=0.$$

7. Prove that the following equality is true

$$\frac{x^2y^2z^2}{a^2b^2}+\frac{(x^2-a^2)(y^2-a^2)(z^2-a^2)}{a^2(a^2-b^2)}+\frac{(x^2-b^2)(y^2-b^2)(z^2-b^2)}{b^2(b^2-a^2)}=$$
$$=x^2+y^2+z^2-a^2-b^2.$$

8. Put

$$\frac{a^k}{(a-b)(a-c)}+\frac{b^k}{(b-a)(b-c)}+\frac{c^k}{(c-a)(c-b)}=S_k.$$

Prove that

$S_0=S_1=0,\ S_2=1,\ S_3=a+b+c,$

$S_4=ab+ac+bc+a^2+b^2+c^2,$

$S_5=a^3+b^3+c^3+a^2b+b^2a+c^2a+a^2c+b^2c+c^2b+abc.$

9. Let

$$\frac{a^k}{(a-b)(a-c)(a-d)}+\frac{b^k}{(b-a)(b-c)(b-d)}+$$
$$+\frac{c^k}{(c-a)(c-b)(c-d)}+\frac{d^k}{(d-a)(d-b)(d-c)}=S_k.$$

Show that

$$S_0=S_1=S_2=0,\quad S_3=1,\quad S_4=a+b+c+d.$$

10. Put

$$\sigma_m=a^m\frac{(a+b)(a+c)}{(a-b)(a-c)}+b^m\frac{(b+c)(b+a)}{(b-c)(b-a)}+c^m\frac{(c+a)(c+b)}{(c-a)(c-b)}.$$

Compute σ_1, σ_2, σ_3 and σ_4.

11. Prove the identity

$$bc\frac{(a-\alpha)(a-\beta)(a-\gamma)}{(a-b)(a-c)}+ca\frac{(b-\alpha)(b-\beta)(b-\gamma)}{(b-a)(b-c)}+$$
$$+ab\frac{(c-\alpha)(c-\beta)(c-\gamma)}{(c-a)(c-b)}=abc-\alpha\beta\gamma.$$

12. Show that

$$\frac{a^2b^2c^2}{(a-d)(b-d)(c-d)}+\frac{a^2b^2d^2}{(a-c)(b-c)(d-c)}+$$
$$+\frac{a^2c^2d^2}{(a-b)(c-b)(d-b)}+\frac{b^2c^2d^2}{(b-a)(c-a)(d-a)}=$$
$$=abc+abd+acd+bcd.$$

13. Simplify the following expressions

1° $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$;

2° $\frac{1}{a^2(a-b)(a-c)}+\frac{1}{b^2(b-a)(b-c)}+\frac{1}{c^2(c-a)(c-b)}$.

14. Simplify the following expression

$$\frac{a^k}{(a-b)(a-c)(x-a)}+\frac{b^k}{(b-a)(b-c)(x-b)}+$$
$$+\frac{c^k}{(c-a)(c-b)(x-c)},$$

where $k=1,\ 2$.

15. Show that

$$\frac{b+c+d}{(b-a)(c-a)(d-a)(x-a)}+\frac{c+d+a}{(c-b)(d-b)(a-b)(x-b)}+$$
$$+\frac{d+a+b}{(d-c)(a-c)(b-c)(x-c)}+\frac{a+b+c}{(a-d)(b-d)(c-d)(x-d)}=$$
$$=\frac{x-a-b-c-d}{(x-a)(x-b)(x-c)(x-d)}.$$

16. Prove the identity

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2.$$

17. Prove the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1.$$

18. Show that if $a+b+c=0$, then

$$\left(\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}\right)\left(\frac{c}{a-b}+\frac{a}{b-c}+\frac{b}{c-a}\right)=9.$$

19. Simplify the following expression

$$\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}.$$

20. Prove that

$$\frac{b-c}{(a-b)(a-c)}+\frac{c-a}{(b-c)(b-a)}+\frac{a-b}{(c-a)(c-b)}=$$
$$=\frac{2}{a-b}+\frac{2}{b-c}+\frac{2}{c-a}.$$

21. Simplify the following expression

$$\frac{a^2-bc}{(a+b)(a+c)}+\frac{b^2-ac}{(b+c)(b+a)}+\frac{c^2-ab}{(c+a)(c+b)}.$$

22. Prove that

$$\frac{d^m(a-b)(b-c)+b^m(a-d)(c-d)}{c^m(a-b)(a-d)+a^m(b-c)(c-d)}=\frac{b-d}{a-c}$$

at $m=1, 2$.

23. Prove that

$$\left\{1-\frac{x}{\alpha_1}+\frac{x(x-\alpha_1)}{\alpha_1\alpha_2}-\frac{x(x-\alpha_1)(x-\alpha_2)}{\alpha_1\alpha_2\alpha_3}+\ldots+\right.$$
$$\left.+(-1)^n\frac{x(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})}{\alpha_1\alpha_2\alpha_3\ldots\alpha_n}\right\}\times$$
$$\times\left\{1+\frac{x}{\alpha_1}+\frac{x(x+\alpha_1)}{\alpha_1\alpha_2}+\frac{x(x+\alpha_1)(x+\alpha_2)}{\alpha_1\alpha_2\alpha_3}+\ldots+\right.$$
$$\left.+\frac{x(x+\alpha_1)(x+\alpha_2)\ldots(x+\alpha_{n-1})}{\alpha_1\alpha_2\alpha_3\ldots\alpha_n}\right\}=$$
$$=1-\frac{x^2}{\alpha_1^2}+\frac{x^2(x^2-\alpha_1^2)}{\alpha_1^2\alpha_2^2}-\ldots+$$
$$+(-1)^n\frac{x^2(x^2-\alpha_1^2)\ldots(x^2-\alpha_{n-1}^2)}{\alpha_1^2\alpha_2^2\ldots\alpha_n^2}$$

24. Given

$$\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ac}+\frac{a^2+b^2-c^2}{2ab}=1.$$

Prove that two of the three fractions must be equal to $+1$, and the third to -1.

25. Show that from the equality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$$

follows

$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}$$

if n is odd.

26. Show that from the equalities

$$\frac{bz+cy}{x(-ax+by+cz)}=\frac{cx+az}{y(ax-by+cz)}=\frac{ay+bx}{z(ax+by-cz)}$$

follows

$$\frac{x}{a(b^2+c^2-a^2)}=\frac{y}{b(a^2+c^2-b^2)}=\frac{z}{c(a^2+b^2-c^2)}.$$

27. Given

$$\alpha+\beta+\gamma=0,$$
$$a+b+c=0,$$
$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0.$$

Prove that

$$\alpha a^2+\beta b^2+\gamma c^2=0.$$

28. If

$$a^3+b^3+c^3=(b+c)(a+c)(a+b)$$

and

$$(b^2+c^2-a^2)x=(c^2+a^2-b^2)y=(a^2+b^2-c^2)z,$$

then

$$x^3+y^3+z^3=(x+y)(x+z)(y+z).$$

29. Consider the finite continued fraction

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots+\cfrac{1}{a_n}}}.$$

Put

$$P_0 = a_0, \quad Q_0 = 1, \quad P_1 = a_0 a_1 + 1, \quad Q_1 = a_1$$

and in general

$$P_{k+1} = a_{k+1} P_k + P_{k-1},$$
$$Q_{k+1} = a_{k+1} Q_k + Q_{k-1}.$$

Then, as is known,

$$\frac{P_n}{Q_n} = a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \qquad (n = 0, 1, 2, 3, \ldots).$$

Prove the following identities

1° $\left(\frac{P_{n+2}}{P_n} - 1\right)\left(1 - \frac{P_{n-1}}{P_{n+1}}\right) = \left(\frac{Q_{n+2}}{Q_n} - 1\right)\left(1 - \frac{Q_{n-1}}{Q_{n+1}}\right);$

2° $\frac{P_n}{Q_n} - \frac{P_0}{Q_0} = \frac{1}{Q_0 Q_1} - \frac{1}{Q_1 Q_2} + \ldots + \frac{(-1)^{n-1}}{Q_{n-1} Q_n};$

3° $P_{n+2} Q_{n-2} - P_{n-2} Q_{n+2} = (a_{n+2} a_{n+1} a_n + a_{n+2} + a_n)(-1)^n;$

4° $\frac{P_n}{P_{n-1}} = a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_0},$

$$\frac{Q_n}{Q_{n-1}} = a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_1}.$$

30. Put for brevity

$$a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_n} = (a_0, a_1, \ldots, a_n) = \frac{P_n}{Q_n},$$

and let the fraction be symmetric, i.e.

$$a_0 = a_n, \quad a_1 = a_{n-1}, \quad \ldots .$$

Prove that

$$P_{n-1} = Q_n.$$

31. Suppose we have a fraction

$$\frac{1}{a} + \frac{1}{a} + \frac{1}{a} + \cdots + \frac{1}{a}.$$

Prove that

$$P_n^2+P_{n+1}^2=P_{n-1}P_{n+1}+P_nP_{n+2}.$$

32. Let

$$x=\cfrac{1}{a+\cfrac{1}{b+\cdots+\cfrac{1}{l+\cfrac{1}{a+\cfrac{1}{b+\cdots+\cfrac{1}{l}}}}}}$$

and let $\frac{P_n}{Q_n}$ and $\frac{P_{n-1}}{Q_{n-1}}$ be, respectively, the last and last but one convergents of the fraction

$$\cfrac{1}{a+\cfrac{1}{b+\cdots+\cfrac{1}{l}}}.$$

Prove that

$$x=\frac{P_nQ_n+P_nP_{n-1}}{Q_n^2+P_nQ_{n-1}}.$$

33. Consider the continued fraction

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cdots+\cfrac{a_n}{b_n}}}.$$

Put

$$P_0=b_0,\quad Q_0=1,\quad P_1=b_0b_1+a_1,\quad Q_1=b_1,\ \ldots$$

and in general

$$P_{k+1}=b_{k+1}P_k+a_{k+1}P_{k-1},$$
$$Q_{k+1}=b_{k+1}Q_k+a_{k+1}Q_{k-1}.$$

Prove that

$$\frac{P_n}{Q_n}=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cdots+\cfrac{a_n}{b_n}}}.$$

34. Prove that

$$\cfrac{r}{r+1-\cfrac{r}{r+1-\cfrac{r}{r+1-\ddots-\cfrac{r}{r+1}}}}=\frac{r^{n+1}-r}{r^{n+1}-1}$$

(the number of links in the continued fraction is equal to n).

35. Prove that

$$\frac{1}{u_1}+\frac{1}{u_2}+\ldots+\frac{1}{u_n}=$$

$$=\cfrac{1}{u_1-\cfrac{u_1^2}{u_1+u_2-\cfrac{u_2^2}{u_2+u_3-\ddots-\cfrac{u_{n-1}^2}{u_{n-1}+u_n}}}}.$$

36. Prove the equality

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots+\cfrac{a_n}{b_n}}}=\cfrac{c_1a_1}{c_1b_1+\cfrac{c_1c_2a_2}{c_2b_2+\ddots+\cfrac{c_{n-1}c_na_n}{c_nb_n}}},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary nonzero quantities.

37. Prove the following identities

1° $$\frac{\sin(n+1)x}{\sin nx}=$$

$$=2\cos x-\cfrac{1}{2\cos x-\cfrac{1}{2\cos x-\ddots-\cfrac{1}{2\cos x}}}$$

(a total of n links);

2° $$1+b_2+b_2b_3+\ldots+b_2b_3\ldots b_n=$$

$$=\cfrac{1}{1-\cfrac{b_2}{b_2+1-\cfrac{b_3}{b_3+1-\ddots-\cfrac{b_n}{b_n+1}}}}.$$

38. Prove that

1° $\sin a+\sin b+\sin c-\sin(a+b+c)=$

$$=4\sin\frac{a+b}{2}\sin\frac{a+c}{2}\sin\frac{b+c}{2};$$

2° $\cos a+\cos b+\cos c+\cos(a+b+c)=$

$$=4\cos\frac{a+b}{2}\cos\frac{b+c}{2}\cos\frac{a+c}{2}.$$

39. Show that

$$\tan a+\tan b+\tan c-\frac{\sin(a+b+c)}{\cos a\cos b\cos c}=\tan a\tan b\tan c.$$

40. Prove that if $A+B+C=\pi$, then we have the following relationships

1° $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;

2° $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$;

3° $\tan A+\tan B+\tan C=\tan A\tan B\tan C$;

4° $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1$;

5° $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

41. Find the algebraic relations between the quantities a, b and c which satisfy the following trigonometric equalities

1° $\cos a+\cos b+\cos c=1+4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}$;

2° $\tan a+\tan b+\tan c=\tan a\tan b\tan c$;

3° $\cos^2 a+\cos^2 b+\cos^2 c-2\cos a\cos b\cos c=1$.

42. Show that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

if

$$xy+xz+yz=1.$$

43. Show that the sum of the three fractions

$$\frac{b-c}{1+bc}, \quad \frac{c-a}{1+ac}, \quad \frac{a-b}{1+ab}$$

is equal to their product.

44. Prove that

$$\tan 3\alpha = \tan\alpha \tan\left(\frac{\pi}{3}+\alpha\right)\tan\left(\frac{\pi}{3}-\alpha\right).$$

45. Prove that from the equality

$$\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$$

follows the relationship

$$\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{1}{(a+b)^3}.$$

46. Suppose we have

$$a_1\cos\alpha_1+a_2\cos\alpha_2+\ldots+a_n\cos\alpha_n=0,$$

$$a_1\cos(\alpha_1+\theta)+a_2\cos(\alpha_2+\theta)+\ldots+a_n\cos(\alpha_n+\theta)=0$$

$$(\theta\neq k\pi).$$

Prove that for any λ

$$a_1\cos(\alpha_1+\lambda)+a_2\cos(\alpha_2+\lambda)+\ldots+a_n\cos(\alpha_n+\lambda)=0.$$

47. Prove the identity

$$\frac{\sin(\beta-\gamma)}{\cos\beta\cos\gamma}+\frac{\sin(\gamma-\alpha)}{\cos\gamma\cos\alpha}+\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=0.$$

48. Let in a triangle the sides be equal to a, b and c, and let

$$r=\frac{s}{p}, \quad r_a=\frac{s}{p-a}, \quad r_b=\frac{s}{p-b}, \quad r_c=\frac{s}{p-c},$$

where s is the area of the triangle and $2p=a+b+c$.

Prove the following relationships

1° $\dfrac{a^2}{r_a-r}+\dfrac{b^2}{r_b-r}+\dfrac{c^2}{r_c-r}=2(r_a+r_b+r_c)$;

2° $\dfrac{a^2r_a}{(a-b)(a-c)}+\dfrac{b^2r_b}{(b-c)(b-a)}+\dfrac{c^2r_c}{(c-a)(c-b)}=\dfrac{p^2}{r}$;

3° $\frac{a+b+c}{r_a+r_b+r_c}\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)=4;$

4° $$\frac{bc}{(a-b)(a-c)r_a^2}+\frac{ac}{(b-c)(b-a)r_b^2}+$$
$$+\frac{ab}{(c-a)(c-b)r_c^2}=\frac{a^2}{(a-b)(a-c)r_br_c}+$$
$$+\frac{b^2}{(b-c)(b-a)r_cr_a}+\frac{c^2}{(c-a)(c-b)r_ar_b}=\frac{1}{r^2};$$

5° $$\frac{ar_a}{(a-b)(a-c)}+\frac{br_b}{(b-c)(b-a)}+\frac{cr_c}{(c-a)(c-b)}=$$
$$=\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+$$
$$+\frac{(a+b)r_c}{(c-a)(c-b)}=\frac{p}{r}.$$

49. Prove the identity

$$\sin(a+b-c-d)=\frac{\sin(a-c)\sin(a-d)}{\sin(a-b)}+$$
$$+\frac{\sin(b-c)\sin(b-d)}{\sin(b-a)}.$$

50. Given

$$\cos\theta=\frac{a}{b+c},\quad \cos\varphi=\frac{b}{a+c},\quad \cos\psi=\frac{c}{a+b}$$

(θ, φ and ψ lie between 0 and π).

Knowing that a, b and c are the sides of a triangle whose angles are A, B and C, correspondingly, prove that

1° $\tan^2\frac{\theta}{2}+\tan^2\frac{\varphi}{2}+\tan^2\frac{\psi}{2}=1;$

2° $\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\tan\frac{\psi}{2}=\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$

51. Prove that

$$\frac{1}{\sin(a-b)\sin(a-c)}+\frac{1}{\sin(b-a)\sin(b-c)}+\frac{1}{\sin(c-a)\sin(c-b)}=\frac{1}{2\cos\frac{a-b}{2}\cos\frac{a-c}{2}\cos\frac{b-c}{2}}.$$

52. Prove the identities

1° $$\frac{\sin a}{\sin(a-b)\sin(a-c)}+\frac{\sin b}{\sin(b-a)\sin(b-c)}+\frac{\sin c}{\sin(c-a)\sin(c-b)}=0;$$

2° $$\frac{\cos a}{\sin(a-b)\sin(a-c)}+\frac{\cos b}{\sin(b-a)\sin(b-c)}+\frac{\cos c}{\sin(c-a)\sin(c-b)}=0.$$

53. Prove the identities

1° $$\sin a\sin(b-c)\cos(b+c-a)+\sin b\sin(c-a)\cos(c+a-b)+\sin c\sin(a-b)\cos(a+b-c)=0;$$

2° $$\cos a\sin(b-c)\sin(b+c-a)+\cos b\sin(c-a)\sin(c+a-b)+\cos c\sin(a-b)\sin(a+b-c)=0;$$

3° $$\sin a\sin(b-c)\sin(b+c-a)+\sin b\sin(c-a)\sin(c+a-b)+\sin c\sin(a-b)\sin(a+b-c)=2\sin(b-c)\sin(c-a)\sin(a-b);$$

4° $$\cos a\sin(b-c)\cos(b+c-a)+\cos b\sin(c-a)\cos(c+a-b)+\cos c\sin(a-b)\cos(a+b-c)=2\sin(b-c)\sin(c-a)\sin(a-b).$$

54. Prove that

1° $\sin^3 A \cos(B-C) + \sin^3 B \cos(C-A) +$
$+ \sin^3 C \cos(A-B) = 3 \sin A \sin B \sin C;$

2° $\sin^3 A \sin(B-C) + \sin^3 B \sin(C-A) +$
$+ \sin^3 C \sin(A-B) = 0$

if $A+B+C=\pi$.

55. Prove the identities

1° $\sin 3A \sin^3(B-C) + \sin 3B \sin^3(C-A) +$
$+ \sin 3C \sin^3(A-B) = 0;$

2° $\sin 3A \cos^3(B-C) + \sin 3B \cos^3(C-A) +$
$+ \sin 3C \cos^3(A-B) = \sin 3A \sin 3B \sin 3C$

if $A+B+C=\pi$.

3. RADICALS. INVERSE TRIGONOMETRIC FUNCTIONS. LOGARITHMS

The symbol $\sqrt[n]{A}$ is understood here (if n is odd) as the only real number whose nth power is equal to A. In this case A can be either less or greater than zero. If n is even, then the symbol $\sqrt[n]{A}$ is understood as the only positive number the nth power of which is equal to A. Here, necessarily, $A \geqslant 0$.

Under these conditions, for instance,

$$\sqrt{A^2} = A \quad \text{if} \quad A > 0,$$
$$\sqrt{A^2} = -A \quad \text{if} \quad A < 0.$$

All the rest of the standard rules and laws governing the operations involving radicals, fractional and negative exponents are considered here to be known. Let us also remind of two formulas which sometimes turn out to be rather useful in performing various transformations, namely:

$$\sqrt{A+\sqrt{B}}=\sqrt{\frac{A+\sqrt{A^2-B}}{2}}+\sqrt{\frac{A-\sqrt{A^2-B}}{2}},$$

$$\sqrt{A-\sqrt{B}}=\sqrt{\frac{A+\sqrt{A^2-B}}{2}}-\sqrt{\frac{A-\sqrt{A^2-B}}{2}}.$$

As far as trigonometric functions are concerned, let us first of all consider the reduction formulas:

1° The functions $\sin x$ and $\cos x$ are characterized by the period 2π, whereas $\tan x$ and $\cot x$ by the period π so that we may write the following equalities

$$\sin(x+2k\pi)=\sin x, \quad \cos(x+2k\pi)=\cos x,$$

$$\tan(x+k\pi)=\tan x, \quad \cot(x+k\pi)=\cot x,$$

where k is any whole number (positive, negative or zero).

2° For the functions $\sin x$ and $\cos x$ the quantity π is the half-period, i.e. the rejection of the quantity $\pm\pi$ in the argument results in a change in the sign of a function. Consequently,

$$\sin(x+k\pi)=(-1)^k \sin x, \quad \cos(x+k\pi)=(-1)^k \cos x,$$

where k is any whole number (positive, negative or zero).

3° The functions $\sin x$, $\tan x$ and $\cot x$ are odd functions, and $\cos x$ is an even function. Therefore

$$\sin(-x)=-\sin x, \quad \tan(-x)=-\tan x,$$

$$\cot(-x)=-\cot x, \quad \cos(-x)=\cos x.$$

4° If x and y are two quantities entering the relationship

$$x+y=\frac{\pi}{2},$$

then

$$\cos x=\sin y, \quad \sin x=\cos y,$$
$$\tan x=\cot y, \quad \cot x=\tan y.$$

Using these remarks, we can always reduce sine or cosine of any argument to sine or cosine of an argument lying in the interval between 0 and $\frac{\pi}{4}$. The same can be said about tangent and cotangent.

Indeed, any argument α can be written in the following form

$$\alpha = s\cdot\frac{\pi}{2} \pm \alpha_0,$$

where s is an integer, and $0 \leqslant \alpha_0 \leqslant \frac{\pi}{4}$, wherefrom follows the stated proposition. Let us also mention the following formulas (k an integer):

$$\sin k\pi = 0, \quad \tan k\pi = 0, \quad \cos k\pi = (-1)^k,$$

$$\sin\frac{k\pi}{2} = 0 \qquad \text{if } k \text{ is even},$$

$$\sin\frac{k\pi}{2} = (-1)^{\frac{k-1}{2}} \qquad \text{if } k \text{ is odd},$$

$$\cos\frac{k\pi}{2} = (-1)^{\frac{k}{2}} \qquad \text{if } k \text{ is even},$$

$$\cos\frac{k\pi}{2} = 0 \qquad \text{if } k \text{ is odd}.$$

Further, we use the symbol arcsin x to denote an arc whose sine is equal to x and which lies in the interval between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Thus, in all cases

$$-\frac{\pi}{2} \leqslant \arcsin x \leqslant +\frac{\pi}{2}.$$

Similarly

$$-\frac{\pi}{2} < \arctan x < +\frac{\pi}{2},$$

$$0 \leqslant \arccos x \leqslant \pi,$$

$$0 < \operatorname{arccot} x < \pi.$$

In this section we also give several problems on transforming expressions containing logarithms.

1. Prove that

$$\left(\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}\right)^2 = 2.$$

2. Show that

1° $\sqrt[3]{\sqrt[3]{2}-1}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$;

2° $\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=\frac{1}{3}(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25})$;

3° $\sqrt{\sqrt[3]{28}-\sqrt[3]{27}}=\frac{1}{3}(\sqrt[3]{98}-\sqrt[3]{28}-1)$;

4° $\left(\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}\right)^{\frac{1}{4}}=\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}$;

5° $\left(\sqrt[5]{\frac{32}{5}}-\sqrt[5]{\frac{27}{5}}\right)^{\frac{1}{3}}=\sqrt[5]{\frac{1}{25}}+\sqrt[5]{\frac{3}{26}}-\sqrt[5]{\frac{9}{25}}$;

6° $\left(\sqrt[5]{\frac{1}{5}}+\sqrt[5]{\frac{4}{5}}\right)^{\frac{1}{2}}=(1+\sqrt[5]{2}+\sqrt{8})^{\frac{1}{5}}=$

$$=\sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}-\sqrt[4]{\frac{1}{125}}.$$

3. Let $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\frac{D}{d}$.

Prove that

$$\sqrt{Aa}+\sqrt{Bb}+\sqrt{Cc}+\sqrt{Dd}=$$
$$=\sqrt{(a+b+c+d)(A+B+C+D)}.$$

4. Show that

$$\sqrt[3]{ax^2+by^2+cz^2}=\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$$

if

$$ax^3=by^3=cz^3 \quad \text{and} \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

5. Put

$$a_n=\left(1+\frac{1}{\sqrt{2}}\right)^n+\left(1-\frac{1}{\sqrt{2}}\right)^n,$$
$$b_n=\left(1+\frac{1}{\sqrt{2}}\right)^n-\left(1-\frac{1}{\sqrt{2}}\right)^n.$$

Show that

$$a_{m+n}=a_m a_n-\frac{a_{m-n}}{2^n},$$

$$b_{m+n}=a_m b_n+\frac{b_{m-n}}{2^n}.$$

6. Let

$$u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]\cdot$$

$$(n=0,\ 1,\ 2,\ 3,\ \ldots).$$

Prove the following relationships

1° $u_{n+1}=u_n+u_{n-1}$;

2° $u_{n-1}=u_k u_{n-k}+u_{k-1}u_{n-k-1}$;

3° $u_{2n-1}=u_n^2+u_{n-1}^2$;

4° $u_{3n}=u_n^3+u_{n+1}^3-u_{n-1}^3$;

5° $u_n^4-u_{n-2}u_{n-1}u_{n+1}u_{n+2}=1$;

6° $u_{n+1}u_{n+2}-u_n u_{n+3}=(-1)^n$;

7° $u_n u_{n+1}-u_{n-2}u_{n-1}=u_{2n-1}$.

7. Prove the following identities

1° $\{2\,[a^2+b^2)^{\frac{1}{2}}-a]\,[(a^2+b^2)^{\frac{1}{2}}-b]\}^{\frac{1}{2}}=$

$$=a+b-(a^2+b^2)^{\frac{1}{2}}\ (a>0,\ b>0);$$

2° $\{3\,[(a^3+b^3)^{\frac{1}{3}}-a]\,[(a^3+b^3)^{\frac{1}{3}}-b]\}^{\frac{1}{3}}=$

$$=(a+b)^{\frac{2}{3}}-(a^2-ab+b^2)^{\frac{1}{3}}.$$

8. Compute the expression

$$(1-ax)(1+ax)^{-1}(1+bx)^{\frac{1}{2}}(1-bx)^{-\frac{1}{2}}$$

at

$$x=a^{-1}\left(2\frac{a}{b}-1\right)^{\frac{1}{2}}\quad (0<a<b<2a).$$

9. Simplify the expression

$$\frac{n^3-3n+(n^2-1)\sqrt{n^2-4}-2}{n^3-3n+(n^2-1)\sqrt{n^2-4}+2}.$$

10. Simplify the expression

$$\left[\frac{\sqrt{1+a}}{\sqrt{1+a}-\sqrt{1-a}}+\frac{1-a}{\sqrt{1-a^2}-1+a}\right]\times$$
$$\times\left[\sqrt{\frac{1}{a^2}-1}-\frac{1}{a}\right] \quad (0<a<1).$$

11. Prove that for $x \geqslant 1$

$$\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$$

is equal to 2 if $x \leqslant 2$, and to $2\sqrt{x-1}$ if $x>2$.

12. Compute

$$\sqrt{a+b+c+2\sqrt{ac+bc}}+\sqrt{a+b+c-2\sqrt{ac+bc}}$$
$$(a \geqslant 0,\ b \geqslant 0,\ c \geqslant 0).$$

13. Prove that the trinomial x^3+px+q vanishes at

$$x=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

14. Express x in terms of a new variable so that $\sqrt{x+a}$ and $\sqrt{x+b}$ become rational.

15. Rationalize the denominator of the fraction

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a'}+\sqrt{b'}+\sqrt{c'}}$$

if

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}.$$

16. Prove that $\sqrt[3]{2}$ cannot be represented in the form $p+\sqrt{q}$, where p and q are rational ($q>0$ and is not a perfect square).

17. Prove the following identities

1° $$\frac{\tan\left(\frac{3\pi}{2}-\alpha\right)\cos\left(\frac{3\pi}{2}-\alpha\right)}{\cos(2\pi-\alpha)}+\cos\left(\alpha-\frac{\pi}{2}\right)\sin(\pi-\alpha)+$$
$$+\cos(\pi+\alpha)\sin\left(\alpha-\frac{\pi}{2}\right)=0;$$

2° $$[1-\sin(3\pi-\alpha)+\cos(3\pi+\alpha)]\times$$
$$\times\left[1-\sin\left(\frac{3\pi}{2}-\alpha\right)+\cos\left(\frac{5\pi}{2}-\alpha\right)\right]+\sin 2\alpha=0;$$

3° $$[1-\sin(\pi+\alpha)+\cos(\pi+\alpha)]^2+$$
$$+\left[1-\sin\left(\frac{3\pi}{2}+\alpha\right)+\right.$$
$$\left.+\cos\left(\frac{3\pi}{2}-\alpha\right)\right]^2=4-2\sin 2\alpha.$$

18. Let $\alpha=2k\pi+\alpha_0$, where $0\leqslant\alpha_0<2\pi$.
Prove that there exists the following equality

$$\sin\frac{\alpha}{2}=(-1)^k\sqrt{\frac{1-\cos\alpha}{2}}.$$

Let us assume then that $\alpha=2k\pi+\alpha_0$, where $-\pi\leqslant\alpha_0<\pi$.
Show that then

$$\cos\frac{\alpha}{2}=(-1)^k\sqrt{\frac{1+\cos\alpha}{2}}.$$

19. If a whole number a is divisible by n leaving no remainder, we shall write this in the following way

$$a\equiv 0 \pmod n$$

which is read: a is comparable with zero by the modulus n. What remainders can a whole number leave when being divided by the whole number n?

It is obvious, that being divided by n, any whole number can leave the following remainders

$$0,\ 1,\ 2,\ 3,\ \ldots,\ n-1.$$

If as a result of dividing a by n we obtain a remainder k, then we shall write

$$a\equiv k \pmod n,$$

since in this case

$$a - k \equiv 0 \pmod{n}.$$

Thus, when dividing a by 2 only two cases are possible: either a is divisible exactly, or leaves a remainder equal to 1. In the first case we write $a \equiv 0 \pmod 2$, in the second $a \equiv 1 \pmod 2$.

The division by 3 can also yield a remainder (0, 1, 2), and, consequently, only three cases are possible: $a \equiv 0 \pmod 3$, $a \equiv 1 \pmod 3$, $a \equiv 2 \pmod 3$ and so on.

Consider the following problem.

We have

$$A_1 = 1.$$

$$A_2 = \cos n\pi.$$

$$A_3 = 2\cos\left(\frac{2}{3}n\pi - \frac{1}{18}\pi\right).$$

$$A_4 = 2\cos\left(\frac{1}{2}n\pi - \frac{1}{8}\pi\right).$$

$$A_5 = 2\cos\left(\frac{2}{5}n\pi - \frac{1}{5}\pi\right) + 2\cos\frac{4}{5}n\pi.$$

$$A_6 = 2\cos\left(\frac{1}{3}n\pi - \frac{5}{18}\pi\right).$$

$$A_7 = 2\cos\left(\frac{2}{7}n\pi - \frac{5}{14}\pi\right) + 2\cos\left(\frac{4}{7}n\pi - \frac{1}{14}\pi\right) + 2\cos\left(\frac{6}{7}n\pi + \frac{1}{14}\pi\right).$$

$$A_8 = 2\cos\left(\frac{1}{4}n\pi - \frac{7}{16}\pi\right) + 2\cos\left(\frac{3}{4}n\pi - \frac{1}{16}\pi\right).$$

$$A_9 = 2\cos\left(\frac{2}{9}n\pi - \frac{14}{27}\pi\right) + 2\cos\left(\frac{4}{9}n\pi - \frac{4}{27}\pi\right) + 2\cos\left(\frac{8}{9}n\pi + \frac{4}{27}\pi\right).$$

$$A_{10} = 2\cos\left(\frac{1}{5}n\pi - \frac{3}{5}\pi\right) + 2\cos\frac{3}{5}n\pi.$$

$$A_{11}=2\cos\left(\frac{2}{11}n\pi-\frac{15}{22}\pi\right)+2\cos\left(\frac{4}{11}n\pi-\frac{5}{22}\pi\right)+$$
$$+2\cos\left(\frac{6}{11}n\pi-\frac{3}{22}\pi\right)+2\cos\left(\frac{8}{11}n\pi-\frac{3}{22}\pi\right)+$$
$$+2\cos\left(\frac{10}{11}n\pi+\frac{5}{22}\pi\right).$$

$$A_{12}=2\cos\left(\frac{1}{6}n\pi-\frac{55}{72}\pi\right)+2\cos\left(\frac{5}{6}n\pi+\frac{1}{72}\pi\right).$$

$$A_{13}=2\cos\left(\frac{2}{13}n\pi-\frac{11}{13}\pi\right)+2\cos\left(\frac{4}{13}n\pi-\frac{4}{13}\pi\right)+$$
$$+2\cos\left(\frac{6}{13}n\pi-\frac{1}{13}\pi\right)+2\cos\left(\frac{8}{13}n\pi+\frac{1}{13}\pi\right)+$$
$$+2\cos\frac{10}{13}n\pi+2\cos\left(\frac{12}{13}n\pi+\frac{4}{13}\pi\right).$$

$$A_{14}=2\cos\left(\frac{1}{7}n\pi-\frac{13}{14}\pi\right)+2\cos\left(\frac{3}{7}n\pi-\frac{3}{14}\pi\right)+$$
$$+2\cos\left(\frac{5}{7}n\pi-\frac{3}{14}\pi\right).$$

$$A_{15}=2\cos\left(\frac{2}{15}n\pi-\frac{1}{90}\pi\right)+2\cos\left(\frac{4}{15}n\pi-\frac{7}{18}\pi\right)+$$
$$+2\cos\left(\frac{8}{15}n\pi-\frac{19}{90}\pi\right)+2\cos\left(\frac{14}{15}n\pi+\frac{7}{18}\pi\right).$$

$$A_{16}=2\cos\left(\frac{1}{8}n\pi+\frac{29}{32}\pi\right)+2\cos\left(\frac{3}{8}n\pi+\frac{27}{32}\pi\right)+$$
$$+2\cos\left(\frac{5}{8}n\pi+\frac{5}{32}\pi\right)+2\cos\left(\frac{7}{8}n\pi+\frac{3}{32}\pi\right).$$

$$A_{17}=2\cos\left(\frac{2}{17}n\pi+\frac{14}{17}\pi\right)+2\cos\left(\frac{4}{17}n\pi-\frac{8}{17}\pi\right)+$$
$$+2\cos\left(\frac{6}{17}n\pi-\frac{5}{17}\pi\right)+2\cos\frac{8}{17}n\pi+$$
$$+2\cos\left(\frac{10}{17}n\pi-\frac{1}{17}\pi\right)+2\cos\left(\frac{12}{17}n\pi-\frac{5}{17}\pi\right)+$$
$$+2\cos\left(\frac{14}{17}n\pi-\frac{1}{17}\pi\right)+2\cos\left(\frac{16}{17}n\pi+\frac{8}{17}\pi\right).$$

$$A_{18}=2\cos\left(\frac{1}{9}n\pi+\frac{20}{27}\pi\right)+2\cos\left(\frac{5}{9}n\pi-\frac{2}{27}\pi\right)+$$
$$+2\cos\left(\frac{7}{9}n\pi+\frac{2}{27}\pi\right).$$

Prove that

$$A_5 = 0 \text{ if } n \equiv 1,\ 2 \pmod 5,$$
$$A_7 = 0 \text{ if } n \equiv 1,\ 3,\ 4 \pmod 7,$$
$$A_{10} = 0 \text{ if } n \equiv 1,\ 2 \pmod 5,$$
$$A_{11} = 0 \text{ if } n \equiv 1,\ 2,\ 3,\ 5,\ 7 \pmod{11},$$
$$A_{13} = 0 \text{ if } n \equiv 2,\ 3,\ 5,\ 7,\ 9,\ 10 \pmod{13},$$
$$A_{14} = 0 \text{ if } n \equiv 1,\ 3,\ 4 \pmod 7,$$
$$A_{16} = 0 \text{ if } n \equiv 0 \pmod 2,$$
$$A_{17} = 0 \text{ if } n \equiv 1,\ 3,\ 4,\ 6,\ 7,\ 9,\ 13,\ 14 \pmod{17},$$

and that A_2, A_3, A_4, A_6, A_8, A_9, A_{12}, A_{15} and A_{18} never vanish for any whole n (S. Ramanujan. *Asymptotic formulae in combinatory analysis*).

20. Let

$$p(n) = A(n+3)^2 + B + C(-1)^n + D\cos\frac{2\pi n}{3} \quad (n \text{ an integer}).$$

Prove that there exists the following relationship

$$p(n) - p(n-1) - p(n-2) + p(n-4) + \\ + p(n-5) - p(n-6) = 0.$$

21. Show that

1° $\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$, $\quad \cos 15° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$;

2° $\sin 18° = \dfrac{-1+\sqrt{5}}{4}$, $\quad \cos 18° = \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$.

22. Show that

$$\sin 6° = \frac{\sqrt{30-6\sqrt{5}} - \sqrt{6+2\sqrt{5}}}{8},$$

$$\cos 6° = \frac{\sqrt{18+6\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{8}.$$

23. Show that

$$\cos(\arcsin x) = \sqrt{1-x^2}, \quad \sin(\arccos x) = \sqrt{1-x^2}.$$

$$\tan(\operatorname{arccot} x) = \frac{1}{x}, \quad \cot(\arctan x) = \frac{1}{x}.$$

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}, \quad \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}.$$

$$\cos(\operatorname{arccot} x) = \frac{x}{\sqrt{1+x^2}}, \quad \sin(\operatorname{arccot} x) = \frac{1}{\sqrt{1+x^2}}.$$

24. Prove that

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}, \quad \arcsin x + \arccos x = \frac{\pi}{2}.$$

25. Prove the equality

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy} + \varepsilon\pi,$$

where $\varepsilon = 0$ if $xy < 1$,
$\varepsilon = -1$ if $xy > 1$ and $x < 0$,
$\varepsilon = +1$ if $xy > 1$ and $x > 0$.

26. Show that $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

27. Show that $\arctan \frac{1}{3} + \arctan \frac{1}{5} + \arctan \frac{1}{7} + \arctan \frac{1}{8} = \frac{\pi}{4}$.

28. Show that $2 \arctan x + \arcsin \frac{2x}{1+x^2} = \pi \quad (x > 1)$.

29. Prove that

$$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2} \quad \text{if } x > 0,$$

$$\arctan x + \arctan \frac{1}{x} = -\frac{\pi}{2} \quad \text{if } x < 0.$$

30. Prove that

$$\arcsin x + \arcsin y = \eta \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) + \varepsilon\pi,$$

where $\eta = 1$, $\varepsilon = 0$ if $xy < 0$ or $x^2 + y^2 \leqslant 1$,
$\eta = -1$, $\varepsilon = -1$ if $x^2 + y^2 > 1$, $x < 0$, $y < 0$,
$\eta = -1$, $\varepsilon = +1$ if $x^2 + y^2 > 1$, $x > 0$, $y > 0$.

31. Check the equality

$$\arccos x + \arccos\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right) = \frac{\pi}{3}$$

if

$$\frac{1}{2} \leqslant x \leqslant 1.$$

32. If

$$A = \arctan\frac{1}{7} \text{ and } B = \arctan\frac{1}{3},$$

then prove that $\cos 2A = \sin 4B$.

33. Let $a^2 + b^2 = 7ab$.

Prove that

$$\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b).$$

34. Prove that $\frac{\log_a n}{\log_{am} n} = 1 + \log_a m$.

35. Prove that from the equalities

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$$

follows $x^y \cdot y^x = z^y \cdot y^z = x^z \cdot z^x$.

36. 1° Prove that $\log_b a \cdot \log_a b = 1$.

2° Simplify the expression

$$a^{\frac{\log(\log a)}{\log a}}$$

(logarithms are taken to one and the same base).

37. Given: $y = 10^{\frac{1}{1-\log x}}$, $z = 10^{\frac{1}{1-\log y}}$ (logarithms are taken to the base 10).

Prove that

$$x = 10^{\frac{1}{1-\log z}}.$$

38. Given

$$a^2 + b^2 = c^2.$$

Prove that

$$\log_{b+c} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a.$$

39. Let $a > 0$, $c > 0$, $b = \sqrt{ac}$, a, c and $ac \neq 1$, $N > 0$. Prove that

$$\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}.$$

40. Prove that

$$\log_{a_1 a_2 \ldots a_n} x = \frac{1}{\frac{1}{\log_{a_1} x} + \frac{1}{\log_{a_2} x} + \ldots + \frac{1}{\log_{a_n} x}}.$$

41. Given a geometric and an arithmetic progression with positive terms

$$a,\ a_1,\ a_2,\ \ldots,\ a_n,\ \ldots;$$

$$b,\ b_1,\ b_2,\ \ldots,\ b_n,\ \ldots.$$

The ratio of the geometric progression and the common difference of the arithmetic progression are positive. Prove that there always exists a system of logarithms for which

$$\log a_n - b_n = \log a - b \quad \text{(for any } n\text{)}.$$

Find the base β of this system.

4. EQUATIONS AND SYSTEMS OF EQUATIONS OF THE FIRST DEGREE

The general form of a first-degree equation in one unknown is

$$Ax + B = 0,$$

where A and B are independent of x. To solve the first-degree equation means to reduce it to this form, since then the expression for the root becomes explicit

$$x = -\frac{B}{A}.$$

Therefore the problem of solving the first-degree equation is one of transforming the given expression to the form $Ax + B = 0$. In doing so great attention should be paid to make sure that all the equations involved are equivalent. The problem of solving a system of equations also consists to a considerable extent in transforming a system into an equivalent one.

This section deals not only with equations of the first degree in the unknown x, but also with the equations which can be reduced to them by means of appropriate transformations (such are equations involving radicals, trigonometric equations and ones involving exponential and logarithmic functions). Here and in the following section we consider a trigonometric equation solved if we find the value of one of the trigonometric functions of an expression linear in x.

Indeed, if it is known that

$$\tan(mx + n) = A,$$

then we find

$$mx + n = \arctan A + k\pi,$$

where k is any integer.

Consequently, all the required values of x are given by formula

$$x = \frac{\arctan A - n + k\pi}{m}.$$

Likewise, if it is found that

$$\cot(mx + n) = A,$$

then

$$mx + n = \operatorname{arccot} A + k\pi \quad \text{and} \quad x = \frac{\operatorname{arccot} A - n + k\pi}{m}.$$

But if it is known that

$$\sin(mx + n) = A,$$

then all the values of x satisfying the last equation are found by the formula

$$mx + n = (-1)^k \arcsin A + k\pi,$$

where k, as before, is any integer.

Analogously, from the equation

$$\cos(mx+n)=A$$

follows

$$mx+n=\pm\arccos A+2k\pi.$$

When solving exponential equations one should remembe that the equation

$$a^x=1 \quad (a>0 \text{ and is not equal to } 1)$$

has the only solution $x=0$.

1. Solve the equation

$$\frac{x-ab}{a+b}+\frac{x-ac}{a+c}+\frac{x-bc}{b+c}=a+b+c.$$

2. Solve the equation

$$\frac{x-a}{bc}+\frac{x-b}{ac}+\frac{x-c}{ab}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

3. Solve the equation

$$\frac{6x+2a+3b+c}{6x+2a-3b-c}=\frac{2x+6a+b+3c}{2x+6a-b-3c}.$$

4. Solve the equation

$$\frac{a+b-x}{c}+\frac{a+c-x}{b}+\frac{b+c-x}{a}+\frac{4x}{a+b+c}=1.$$

5. Solve the equation

$$\frac{\sqrt[p]{b+x}}{b}+\frac{\sqrt[p]{b+x}}{x}=\frac{c\sqrt[p]{x}}{a}.$$

6. Solve the equations

1° $\sqrt{x+1}+\sqrt{x-1}=1$;

2° $\sqrt{x+1}-\sqrt{x-1}=1$

7. Solve the equation

$$\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}}=\sqrt[3]{b}.$$

8. Solve the equation

$$\sqrt{1-\sqrt{x^4-x^2}}=x-1.$$

9. Solve the equation

$$\frac{\sqrt{a}+\sqrt{x-b}}{\sqrt{a}+\sqrt{x-a}}=\sqrt{\frac{a}{b}}.$$

10. Solve the equation

$$\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{b} \quad (a>0).$$

11. Solve the system

$$\begin{aligned} x+y+z&=a\\ x+y+v&=b\\ x+z+v&=c\\ y+z+v&=d. \end{aligned}$$

12. Solve the system

$$\begin{aligned} x_1+x_2+x_3+x_4&=2a_1\\ x_1+x_2-x_3-x_4&=2a_2\\ x_1-x_2+x_3-x_4&=2a_3\\ x_1-x_2-x_3+x_4&=2a_4. \end{aligned}$$

13. Solve the system

$$\begin{aligned} ax+m(y+z+v)&=k\\ by+m(x+z+v)&=l\\ cz+m(x+y+v)&=p\\ dv+m(x+y+z)&=q. \end{aligned}$$

14. Solve the system

$$\frac{x_1-a_1}{m_1}=\frac{x_2-a_2}{m_2}=\ldots=\frac{x_p-a_p}{m_p}$$

$$x_1+x_2+\ldots+x_p=a.$$

15. Solve the system

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=a$$
$$\frac{1}{v}+\frac{1}{x}+\frac{1}{y}=b$$
$$\frac{1}{v}+\frac{1}{z}+\frac{1}{x}=c$$
$$\frac{1}{y}+\frac{1}{z}+\frac{1}{v}=d.$$

16. Solve the system

$$ay + bx = c$$
$$cx + az = b$$
$$bz + cy = a.$$

17. Solve the system

$$cy + bz = 2dyz$$
$$az + cx = 2d'zx$$
$$bx + ay = 2d''xy.$$

18. Solve the system

$$\frac{xy}{ay+bx}=c, \qquad \frac{xz}{az+cx}=b, \qquad \frac{yz}{bz+cy}=a.$$

19. Solve the system

$$y+z-x=\frac{xyz}{a^2}$$
$$z+x-y=\frac{xyz}{b^2}$$
$$x+y-z=\frac{xyz}{c^2}.$$

20. Solve the system

$$(b + c)(y + z) - ax = b - c$$
$$(c + a)(x + z) - by = c - a$$
$$(a + b)(x + y) - cz = a - b$$

if

$$a + b + c \neq 0.$$

21. Solve the system

$$(c+a)\,y+(a+b)\,z-(b+c)\,x=2a^3$$
$$(a+b)\,z+(b+c)\,x-(c+a)\,y=2b^3$$
$$(b+c)\,x+(c+a)\,y-(a+b)\,z=2c^3$$

if

$$b+c\neq 0,\ a+c\neq 0,\ a+b\neq 0$$

22. Solve the system

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1$$
$$\frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}=1$$
$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}=1.$$

23. Solve the system

$$z+ay+a^2x+a^3=0$$
$$z+by+b^2x+b^3=0$$
$$z+cy+c^2x+c^3=0.$$

24. Solve the system

$$z+ay+a^2x+a^3t+a^4=0$$
$$z+by+b^2x+b^3t+b^4=0$$
$$z+cy+c^2x+c^3t+c^4=0$$
$$z+dy+d^2x+d^3t+d^4=0.$$

25. Solve the system

$$x+y+z+u=m$$
$$ax+by+cz+du=n$$
$$a^2x+b^2y+c^2z+d^2u=k$$
$$a^3x+b^3y+c^3z+d^3u=l.$$

26. Solve the system

$$x_1+2x_2+3x_3+\ldots+nx_n=a_1$$
$$x_2+2x_3+3x_4+\ldots+nx_1=a_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n+2x_1+3x_2+\ldots+nx_{n-1}=a_n.$$

27. Solve the system

$$\begin{aligned} x_1 - x_2 - x_3 - \dots - x_n &= 2a \\ -x_1 + 3x_2 - x_3 - \dots - x_n &= 4a \\ -x_1 - x_2 + 7x_3 - \dots - x_n &= 8a \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ -x_1 - x_2 - x_3 - \dots + (2^n - 1)x_n &= 2^n a. \end{aligned}$$

28. Solve the system

$$\begin{aligned} x_1 + x_2 + x_3 + \dots + x_n &= 1 \\ x_1 + x_3 + \dots + x_n &= 2 \\ x_1 + x_2 + x_4 + \dots + x_n &= 3 \\ \dots\dots\dots\dots\dots\dots \\ x_1 + x_2 + \dots + x_{n-1} &= n. \end{aligned}$$

29. Show that for the equations

$$ax + b = 0, \quad a'x + b' = 0.$$

to be compatible it is necessary and sufficient that

$$ab' - a'b = 0.$$

30. Show that the systems

$$\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}$$

and

$$\begin{aligned} l(ax + by + c) + l'(a'x + b'y + c) &= 0 \\ m(ax + by + c) + m'(a'x + b'y + c') &= 0 \end{aligned}$$

are equivalent if

$$lm' - l'm \neq 0.$$

31. Prove that the system

$$\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}$$

has one and only one solution if

$$ab' - a'b \neq 0.$$

32. Prove that from the equations

$$ax+by=0$$
$$a'x+b'y=0,$$

if $ab'-a'b\neq 0$, follows

$$x=y=0.$$

33. Show that the following three equations are compatible

$$ax+by+c=0,$$
$$a'x+b'y+c'=0,$$
$$a''x+b''y+c''=0$$

if $a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)=0.$

34. Let a, b, c be distinct numbers. Prove that from the equations:

$$x+ay+a^2z=0,$$
$$x+by+b^2z=0,$$
$$x+cy+c^2z=0$$

follows

$$x=y=z=0.$$

35. Prove that from the equations

$$Ax+By+Cz=0,$$
$$A_1x+B_1y+C_1z=0$$

follows

$$\frac{x}{C_1B-CB_1}=\frac{y}{CA_1-C_1A}=\frac{z}{AB_1-A_1B}$$

if not all of the denominators are equal to zero.

36. Prove that the elimination of x, y, z from the equations

$$ax+cy+bz=0,$$
$$cx+by+az=0,$$
$$bx+ay+cz=0$$

yields

$$a^3+b^3+c^3-3abc=0.$$

37. Given the system

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$$
$$\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$$
$$\frac{x}{a}+\frac{z}{c}=\mu\left(1-\frac{y}{b}\right)$$
$$\frac{x}{a}-\frac{z}{c}=\frac{1}{\mu}\left(1+\frac{y}{b}\right).$$

Prove that the equations are compatible and determine x, y and z.

38. Determine whether the equations of the system

$$(a+b)x+(ap+bq)y=ap^2+bq^2$$
$$(ap+bq)x+(ap^2+bq^2)y=ap^3+bq^3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(ap^{k-1}+bq^{k-1})x+(ap^k+bq^k)y=ap^{k+1}+bq^{k+1}$$

are compatible.

39. Solve the system

$$x_1+x_2=a_1$$
$$x_2+x_3=a_2$$
$$x_3+x_4=a_3$$
$$\cdots\cdots\cdots$$
$$x_{n-1}+x_n=a_{n-1}$$
$$x_n+x_1=a_n$$

40. Solve the system

$$x+y+z=0$$
$$\frac{a^2x}{a-d}+\frac{b^2y}{b-d}+\frac{c^2z}{c-d}=0$$
$$\frac{ax}{a-d}+\frac{by}{b-d}+\frac{cz}{c-d}=d(a-b)(b-c)(c-a).$$

41. Solve the system

$$(x + a)(y + l) = (a - n)(l - b)$$
$$(y + b)(z + m) = (b - l)(m - c)$$
$$(z + c)(x + n) = (c - m)(n - a).$$

42. Determine k for the system

$$x + (1 + k)y = 0$$
$$(1 - k)x + ky = 1 + k$$
$$(1 + k)x + (12 - k)y = -(1 + k)$$

to be compatible.

43. Solve the system

$$x \sin a + y \sin 2a + z \sin 3a = \sin 4a$$
$$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$$
$$x \sin c + y \sin 2c + z \sin 3c = \sin 4c.$$

44. Show that from the equalities

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad A + B + C = \pi$$

follows

$$a = b \cos C + c \cos B,$$
$$b = c \cos A + a \cos C,$$
$$c = a \cos B + b \cos A.$$

45. Show that from the given data

$$a = b \cos C + c \cos B,$$
$$b = c \cos A + a \cos C,$$
$$c = a \cos B + b \cos A,$$

$$0 < A < \pi, \quad 0 < B < \pi, \quad 0 < C < \pi, \quad a > 0,$$
$$b > 0, \quad c > 0,$$

follows

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{and} \quad A + B + C = \pi.$$

46. Given

$$\begin{aligned} a &= b\cos C + c\cos B & a^2 &= b^2 + c^2 - 2bc\cos A \\ b &= c\cos A + a\cos C \quad (1) & b^2 &= a^2 + c^2 - 2ac\cos B \quad (2) \\ c &= a\cos B + b\cos A & c^2 &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

Show that systems (1) and (2) are equivalent, i.e. from equations (1) follow equations (2) and, conversely, from equations (2) follow equations (1).

47. Given

$$\begin{aligned} \cos a &= \cos b\cos c + \sin b\sin c\cos A, \\ \cos b &= \cos a\cos c + \sin a\sin c\cos B, \qquad (*) \\ \cos c &= \cos a\cos b + \sin a\sin b\cos C, \end{aligned}$$

where a, b, c and A, B, C are between 0 and π.

Prove that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

48. Prove that from the conditions of tne preceding problem follows

1° $$\begin{aligned} \cos A &= -\cos B\cos C + \sin B\sin C\cos a, \\ \cos B &= -\cos A\cos C + \sin A\sin C\cos b, \\ \cos C &= -\cos A\cos B + \sin A\sin B\cos c; \end{aligned}$$

2° $$\tan\frac{1}{4}\varepsilon = \sqrt{\tan\frac{p}{2}\tan\frac{p-a}{2}\tan\frac{p-b}{2}\tan\frac{p-c}{2}}$$

if $\varepsilon = A + B + C - \pi$ and $2p = a + b + c$.

49. Solve the equation

$$(b - c)\tan(x + \alpha) + (c - a)\tan(x + \beta) + (a - b)\tan(x + \gamma) = 0.$$

50. Prove that $\sin x$ and $\cos x$ are rational if and only if $\tan\frac{x}{2}$ is rational.

51. Solve the equation

$$\sin^4 x + \cos^4 x = a.$$

52. Solve the following equations

1° $\sin x + \sin 2x + \sin 3x = 0$;

2° $\cos nx + \cos (n - 2) x - \cos x = 0$.

53. Solve the equation

1° $m \sin (a - x) = n \sin (b - x)$;

2° $\sin (x + 3\alpha) = 3 \sin (\alpha - x)$.

54. Solve the equation

$$\sin 5x = 16 \sin^5 x.$$

55. Solve the equation

$$\sin x + 2 \sin x \cos (a - x) = \sin a.$$

56. Solve the equation

$$\sin x \sin (\gamma - x) = a.$$

57. Solve the equation

$$\sin (\alpha + x) + \sin \alpha \sin x \tan (\alpha + x) = m \cos \alpha \cos x.$$

58. Solve the equation

$$\cos^2 \alpha + \cos^2 x + \cos^2 (\alpha + x) = 1 + 2 \cos \alpha \cos (\alpha+x)$$

59. Solve the equation

$$(1 - \tan x)(1 + \sin 2x) = 1 + \tan x.$$

60. Show that if

$$\tan x + \tan 2x + \tan 3x + \tan 4x = 0,$$

then either $5x = k\pi$, or $8 \cos 2x = 1 \pm \sqrt{17}$.

61. Given the expression

$$ax^2 + 2bxy + cy^2.$$

Make the substitution

$$x = X \cos \theta - Y \sin \theta,$$
$$y = X \sin \theta + Y \cos \theta.$$

It is required to choose the angle θ so that to ensure the identity

$$ax^2 + 2bxy + cy^2 = AX^2 + BY^2.$$

62. Show that from the equalities

$$\frac{x}{\tan(\theta+\alpha)} = \frac{y}{\tan(\theta+\beta)} = \frac{z}{\tan(\theta+\gamma)}$$

follows

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta) + \frac{y+z}{y-z}\sin^2(\beta-\gamma) + \frac{z+x}{z-x}\sin^2(\gamma-\alpha) = 0.$$

63. Solve the systems

1° $\frac{\sin x}{a} = \frac{\sin y}{b} = \frac{\sin z}{c}$

$x + y + z = \pi$;

2° $\frac{\tan x}{a} = \frac{\tan y}{b} = \frac{\tan z}{c}$

$x + y + z = \pi$.

64. Solve the system

$$\tan x \tan y = a$$
$$x + y = 2b.$$

65. Solve the equation

$$4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}.$$

66. Find the positive solutions of the equation

$$x^{x+1} = 1.$$

67. Solve the system

$$a^x b^y = m$$
$$x + y = n\ (a > 0,\ b > 0).$$

68. Solve the system

$$x^y = y^x$$
$$a^x = b^y.$$

69. Solve the system

$$(ax)^{\log a} = (by)^{\log b}$$
$$b^{\log x} = a^{\log y}.$$

70. Solve the system

$$x^y = y^x$$
$$x^m = y^n.$$

5. EQUATIONS AND SYSTEMS OF EQUATIONS OF THE SECOND DEGREE

The present section contains mainly problems on solving quadratic equations and using the properties of the second-degree trinomial.

It should be remembered that if the roots of the trinomial $ax^2 + bx + c$* are imaginary, then this trinomial retains its sign at any real values of x. As is easily seen in this case the sign of the trinomial coincides with that of the constant term (i.e. with the sign of c). Thus, if $c > 0$ and the roots of the trinomial $ax^2 + bx + c$ are imaginary, then

$$ax^2 + bx + c > 0$$

for any real x.

When solving systems of equations the following proposition should be taken into account. Let a system of m equations in m unknowns be under consideration, the degrees of these equations being, respectively,

$$k_1, k_2, \ldots, k_m.$$

Then our system, generally speaking, allows for $k_1 k_2 \ldots k_m$ solution sets. To be more precise, the product of the degrees of the equations is the maximal number of solutions. Sometimes this limit is reached (see Problem 23), but sometimes it is not. Nevertheless, this proposition is of importance, since it prevents the loss of solutions.

1. Solve the equation

$$x^2 \frac{(b+x)(x+c)}{(x-b)(x-c)} + b^2 \frac{(b+c)(b+x)}{(b-c)(b-x)} + c^2 \frac{(c+x)(c+b)}{(c-x)(c-b)} = (b+c)^2.$$

---

* In this section the letters a, b, c, p, q and other constants in the equations denote real numbers.

2. Solve the equation

$$a^3(b-c)(x-b)(x-c)+b^3(c-a)(x-c)(x-a)+$$
$$+c^3(a-b)(x-a)(x-b)=0$$

and show that if the roots of this equation are equal, then exists one of the following equalities

$$\frac{1}{\sqrt{a}} \pm \frac{1}{\sqrt{b}} \pm \frac{1}{\sqrt{c}} = 0.$$

3. Solve the equation

$$\frac{(a-x)\sqrt{a-x}-(b-x)\sqrt{x-b}}{\sqrt{a-x}+\sqrt{x-b}} = a-b.$$

4. Solve the equation

$$\sqrt{4a+b-5x}+\sqrt{4b+a-5x}-3\sqrt{a+b-2x}=0.$$

5. Prove that the roots of the equation

$$(x-a)(x-c)+\lambda(x-b)(x-d)=0$$

are real for any λ if $a<b<c<d$.

6. Show that the roots of the equation

$$(x-a)(x-b)+(x-a)(x-c)+(x-b)(x-c)=0$$

are always real.

7. Prove that at least one of the equations

$$x^2+px+q=0,$$
$$x^2+p_1x+q_1=0$$

has real roots if $p_1p=2(q_1+q)$.

8. Prove that the roots of the equation

$$a(x-b)(x-c)+b(x-a)(x-c)+$$
$$+c(x-a)(x-b)=0$$

are always real.

9. Find the values of p and q for which the roots of the equation

$$x^2+px+q=0$$

are equal to p and q.

10. Prove that for any real x, y and z there exists the following inequality

$$x^2+y^2+z^2-xy-xz-yz \geqslant 0.$$

11. Let
$$x + y + z = a.$$
Show that then
$$x^2 + y^2 + z^2 \geqslant \frac{a^2}{3}.$$
12. Prove the inequality
$$x + y + z \leqslant \sqrt{3(x^2 + y^2 + z^2)}.$$
13. Let α and β be the roots of the quadratic equation
$$x^2 + px + q = 0.$$
Put $\alpha^k + \beta^k = s_k$.

Express s_k in terms of p and q at $k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

14. Let α and β be the roots of the quadratic equation
$$x^2 + px + q = 0 \quad (\alpha > 0,\ \beta > 0).$$
Express $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$ in terms of the coefficients of the equation.

15. Show that if the two equations
$$Ax^2 + Bx + C = 0, \quad A'x^2 + B'x + C' = 0$$
have a common root, then
$$(AC' - CA')^2 = (AB' - BA')(BC' - CB').$$
16. Solve the system
$$x(x + y + z) = a^2$$
$$y(x + y + z) = b^2$$
$$z(x + y + z) = c^2.$$
17. Solve the system
$$x(x + y + z) = a - yz$$
$$y(x + y + z) = b - xz$$
$$z(x + y + z) = c - xy.$$
18. Solve the system
$$y + 2x + z = a(y + x)(z + x)$$
$$z + 2y + x = b(z + y)(x + y)$$
$$x + 2z + y = c(y + z)(x + z).$$

19. Solve the system

$$\begin{aligned} y + z + yz &= a \\ x + z + xz &= b \\ x + y + xy &= c. \end{aligned}$$

20. Solve the system

$$\begin{aligned} yz &= ax \\ zx &= by \quad (a > 0,\ b > 0,\ c > 0). \\ xy &= cz \end{aligned}$$

21. Solve the system

$$\begin{aligned} x^2 + y^2 &= cxyz \\ x^2 + z^2 &= bxyz \\ y^2 + z^2 &= axyz. \end{aligned}$$

22. Solve the system

$$\begin{aligned} x(y + z) &= a^2 \\ y(x + z) &= b^2 \\ z(x + y) &= c^2. \end{aligned}$$

23. Solve the system

$$\begin{aligned} x^3 &= ax + by \\ y^3 &= bx + ay. \end{aligned}$$

24. Solve the system

$$\begin{aligned} x^2 &= a + (y - z)^2 \\ y^2 &= b + (x - z)^2 \\ z^2 &= c + (x - y)^2. \end{aligned}$$

25. Solve the system

$$\begin{aligned} \frac{b(x+y)}{x+y+cxy} + \frac{c(z+x)}{x+z+bxz} &= a \\ \frac{c(y+z)}{y+z+ayz} + \frac{a(x+y)}{x+y+cxy} &= b \\ \frac{a(x+z)}{x+z+bxz} + \frac{b(y+z)}{y+z+ayz} &= c. \end{aligned}$$

26. Solve the system

$$x^2 - yz = a$$
$$y^2 - xz = b$$
$$z^2 - xy = c.$$

27. Solve the system

$$y^2 + z^2 - (y + z)\,x = a$$
$$x^2 + z^2 - (x + z)\,y = b$$
$$x^2 + y^2 - (x + y)\,z = c.$$

28. Solve the system

$$x^2 + y^2 + xy = c^2$$
$$z^2 + x^2 + xz = b^2$$
$$y^2 + z^2 + yz = a^2.$$

29. Solve the system

$$x^3 + y^3 + z^3 = a^3$$
$$x^2 + y^2 + z^2 = a^2$$
$$x + y + z = a.$$

30. Solve the system

$$x^4 + y^4 + z^4 + u^4 = a^4$$
$$x^3 + y^3 + z^3 + u^3 = a^3$$
$$x^2 + y^2 + z^2 + u^2 = a^2$$
$$x + y + z + u = a.$$

31. Prove that systems of equalities (1) and (2) are equivalent, i.e. from existence of (1) follows the existence of (2) and conversely.

$$\begin{aligned} a^2+b^2+c^2&=1, & aa'+bb'+cc'&=0,\\ a'^2+b'^2+c'^2&=1, & a'a''+b'b''+c'c''&=0,\\ a''^2+b''^2+c''^2&=1, & aa''+bb''+cc''&=0; \end{aligned} \qquad (1)$$

$$\begin{aligned} a^2+a'^2+a''^2&=1, & ab+a'b'+a''b''&=0,\\ b^2+b'^2+b''^2&=1, & bc+b'c'+b''c''&=0,\\ c^2+c'^2+c''^2&=1, & ca+c'a'+c''a''&=0. \end{aligned} \qquad (2)$$

32. Eliminate x, y and z from the equalities

$$x^2(y+z)=a^3, \quad y^2(x+z)=b^3, \quad z^2(x+y)=c^3, \quad xyz=abc.$$

33. Given

$$\frac{y}{z}-\frac{z}{y}=a, \quad \frac{z}{x}-\frac{x}{z}=b, \quad \frac{x}{y}-\frac{y}{x}=c.$$

Eliminate x, y and z.

34. Eliminate x, y, z from the system

$$\begin{aligned} y^2+z^2-2ayz&=0\\ z^2+x^2-2bxz&=0\\ x^2+y^2-2cxy&=0. \end{aligned}$$

35. Show that the elimination of x, y and z from the system

$$\begin{aligned} y^2+yz+z^2&=a^2\\ z^2+xz+x^2&=b^2\\ x^2+xy+y^2&=c^2\\ xy+yz+xz&=0 \end{aligned}$$

yields

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)=0.$$

36. Eliminate x and y from the equations

$$x+y=a, \quad x^2+y^2=b, \quad x^3+y^3=c.$$

37. Eliminate a, b, c from the system

$$\begin{aligned} \frac{x}{a}=\frac{y}{b}&=\frac{z}{c}\\ a^2+b^2+c^2&=1\\ a+b+c&=1. \end{aligned}$$

38. Given

$$\begin{aligned} \frac{x}{y}+\frac{y}{z}+\frac{z}{x}&=\alpha\\ \frac{x}{z}+\frac{y}{x}+\frac{z}{y}&=\beta\\ \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)&=\gamma. \end{aligned}$$

Eliminate x, y and z.

39. Prove that if

$$x + y + z + w = 0$$

$$ax + by + cz + dw = 0$$

$$(a-d)^2(b-c)^2(xw+yz) + (b-d)^2(c-a)^2(yw+zx) + \\ + (c-d)^2(a-b)^2(zw+xy) = 0,$$

then

$$\frac{x}{(d-b)(d-c)(b-c)} = \frac{y}{(d-c)(d-a)(c-a)} = \\ = \frac{z}{(d-a)(d-b)(a-b)} = \frac{w}{(b-c)(c-a)(a-b)}.$$

40. 1° Let

$$0 < \alpha < \pi, \quad 0 < \beta < \pi$$

and

$$\cos\alpha + \cos\beta - \cos(\alpha+\beta) = \frac{3}{2}.$$

Prove that

$$\alpha = \beta = \frac{\pi}{3}.$$

2° Let

$$0 < \alpha < \pi, \quad 0 < \beta < \pi$$

and

$$\cos\alpha\cos\beta\cos(\alpha+\beta) = -\frac{1}{8}.$$

Prove that

$$\alpha = \beta = \frac{\pi}{3}.$$

41. Let

$$\cos\theta + \cos\varphi = a, \quad \sin\theta + \sin\varphi = b.$$

Compute

$$\cos(\theta+\varphi) \quad \text{and} \quad \sin(\theta+\varphi).$$

42. Given that α and β are different solutions of the equation

$$a\cos x + b\sin x = c.$$

Prove that

$$\cos^2\frac{\alpha-\beta}{2}=\frac{c^2}{a^2+b^2}.$$

43. Let

$$\frac{\sin(\theta-\alpha)}{\sin(\theta-\beta)}=\frac{a}{b},\qquad \frac{\cos(\theta-\alpha)}{\cos(\theta-\beta)}=\frac{c}{d}.$$

Prove that

$$\cos(\alpha-\beta)=\frac{ac+bd}{ad+bc}.$$

44. Given

$$\frac{e^2-1}{1+2e\cos\alpha+e^2}=\frac{1+2e\cos\beta+e^2}{e^2-1}.$$

Prove that

1° $\dfrac{e^2-1}{1+2e\cos\alpha+e^2}=\dfrac{e+\cos\beta}{e+\cos\alpha}=\pm\dfrac{\sin\beta}{\sin\alpha}=-\dfrac{1+e\cos\beta}{1+e\cos\alpha}$;

2° $\tan\dfrac{\alpha}{2}\cdot\tan\dfrac{\beta}{2}=\pm\dfrac{1+e}{1-e}$.

45. Prove that if

$$\frac{\cos x-\cos\alpha}{\cos x-\cos\beta}=\frac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha},$$

then one of the values of $\tan\frac{x}{2}$ is $\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}$.

46. Let

$$\cos\alpha=\cos\beta\cos\varphi=\cos\gamma\cos\theta,\quad \sin\alpha=2\sin\frac{\varphi}{2}\sin\frac{\theta}{2}.$$

Prove that

$$\tan^2\frac{\alpha}{2}=\tan^2\frac{\beta}{2}\cdot\tan^2\frac{\gamma}{2}.$$

47. Show that if

$$(x-a)\cos\theta+y\sin\theta=(x-a)\cos\theta_1+y\sin\theta_1=a$$

and

$$\tan\frac{\theta}{2}-\tan\frac{\theta_1}{2}=2l,$$

then

$$y^2=2ax-(1-l^2)x^2.$$

48. Prove that from the equalities

$$x\cos\theta + y\sin\theta = x\cos\varphi + y\sin\varphi = 2a$$

and

$$2\sin\frac{\theta}{2}\sin\frac{\varphi}{2} = 1$$

follows

$$y^2 = 4a(a - x).$$

49. Let

$$\cos\theta = \cos\alpha\cos\beta.$$

Prove that

$$\tan\frac{\theta+\alpha}{2}\cdot\tan\frac{\theta-\alpha}{2} = \tan^2\frac{\beta}{2}.$$

50. Show that if

$$\frac{\cos x}{a} = \frac{\cos(x+\theta)}{b} = \frac{\cos(x+2\theta)}{c} = \frac{\cos(x+3\theta)}{d},$$

then

$$\frac{a+c}{b} = \frac{b+d}{c}.$$

51. Let

$$\cos^2\theta = \frac{\cos\alpha}{\cos\beta}, \quad \cos^2\varphi = \frac{\cos\gamma}{\cos\beta}, \quad \frac{\tan\theta}{\tan\varphi} = \frac{\tan\alpha}{\tan\gamma}.$$

Prove that

$$\tan^2\frac{\alpha}{2}\cdot\tan^2\frac{\gamma}{2} = \tan^2\frac{\beta}{2}.$$

52. Prove that if

$$\cos\theta = \cos\alpha\cos\beta, \quad \cos\varphi = \cos\alpha_1\cos\beta, \quad \tan\frac{\theta}{2}\tan\frac{\varphi}{2} = \tan\frac{\beta}{2},$$

then

$$\sin^2\beta = \left(\frac{1}{\cos\alpha} - 1\right)\left(\frac{1}{\cos\alpha_1} - 1\right).$$

53. Let

$$x\cos(\alpha+\beta) + \cos(\alpha-\beta) = x\cos(\beta+\gamma) + \cos(\beta-\gamma) =$$
$$= x\cos(\gamma+\alpha) + \cos(\gamma-\alpha).$$

Prove that

$$\frac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)}=\frac{\tan\beta}{\tan\frac{1}{2}(\alpha+\gamma)}=\frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}.$$

54. Prove that if

$$\frac{\sin(\theta-\beta)\cos\alpha}{\sin(\varphi-\alpha)\cos\beta}+\frac{\cos(\alpha+\theta)\sin\beta}{\cos(\varphi-\beta)\sin\alpha}=0$$

and

$$\frac{\tan\theta\tan\alpha}{\tan\varphi\tan\beta}+\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}=0,$$

then

$$\tan\theta=\frac{1}{2}(\tan\beta+\cot\alpha),\quad \tan\varphi=\frac{1}{2}(\tan\alpha-\cot\beta).$$

55. Given

$$n^2\sin^2(\alpha+\beta)=\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos(\alpha-\beta).$$

Prove that

$$\tan\alpha=\frac{1\pm n}{1\mp n}\tan\beta.$$

56. Eliminate θ from the equations

$$\cos(\alpha-3\theta)=m\cos^3\theta,\quad \sin(\alpha-3\theta)=m\sin^3\theta.$$

57. Eliminate θ from the equations

$$(a-b)\sin(\theta+\varphi)=(a+b)\sin(\theta-\varphi),$$

$$a\tan\frac{\theta}{2}-b\tan\frac{\varphi}{2}=c.$$

58. Show that the result of elimination of θ and φ from the equations

$$\cos\theta=\frac{\sin\beta}{\sin\alpha},\quad \cos\varphi=\frac{\sin\gamma}{\sin\alpha},\quad \cos(\theta-\varphi)=\sin\beta\sin\gamma$$

is

$$\tan^2\alpha=\tan^2\beta+\tan^2\gamma.$$

59. Eliminate θ and φ from the equations

$$a\sin^2\theta+b\cos^2\theta=a\cos^2\varphi+b\sin^2\varphi=1,$$

$$a\tan\theta=b\tan\varphi.$$

60. Prove that if

$$\cos(\theta-\alpha)=a,\quad \sin(\theta-\beta)=b,$$

then

$$a^2 - 2ab \sin(\alpha - \beta) + b^2 = \cos^2(\alpha - \beta).$$

61. Solve the equation

$$\cos 3x \cos^3 x + \sin 3x \sin^3 x = 0.$$

62. Solve the equation

$$\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0.$$

63. Solve the equation

$$\tan^2 x = \frac{1-\cos x}{1-\sin x}.$$

64. Solve the equation

$$32 \cos^6 x - \cos 6x = 1.$$

65. Solve and analyze the equation

$$\sin 3x + \sin 2x = m \sin x.$$

66. Solve the equation

$$(1+k)\frac{\cos x \cos(2x-\alpha)}{\cos(x-\alpha)} = 1 + k \cos 2x.$$

67. Solve the equation

$$\sin^4 x + \cos^4 x - 2 \sin 2x + \frac{3}{4} \sin^2 2x = 0.$$

68. Solve the equation

$$2 \log_x a + \log_{ax} a + 3 \log_{a^2x} a = 0.$$

69. Find the positive solutions of the system

$$x^{x+y} = y^a, \quad y^{x+y} = x^{4a} \quad (a > 0).$$

70. Find the positive values of the unknowns x, y, u and v satisfying the system

$$u^p v^q = a^x, \quad u^q v^p = a^y, \quad u^x v^y = b, \quad u^y v^x = c$$

$$(a,\ b,\ c > 0 \text{ and } p^2 - q^2 \neq 0).$$

6. COMPLEX NUMBERS AND POLYNOMIALS

We proceed here from the assumption that the principal operations with complex numbers (i.e. addition, multiplication, division and evolution) are already known to the reader. Likewise, we take as known the trigonometric form of a complex number and de Moivre's formula. In factoring polynomials and solving certain higher-degree equations an important role is played by the so-called remainder theorem (stated by the French mathematician Bézout), usually considered in textbooks of elementary algebra. Let us recall it: if $f(x)$ is a polynomial in x and if $f(a) = 0$, then $f(x)$ is exactly divisible by $x - a$. Hence (assuming that the polynomial has one root) follows the possibility of resolving an nth-degree polynomial into n, equal or unequal, linear factors as well as the following proposition used here repeatedly: if it is known that a certain nth-degree polynomial in x vanishes at $n + 1$ different values of x, then such a polynomial identically equals zero. Consequently, if two polynomials of the nth degree ni x attain equal values at $n + 1$ different values of x, then such polynomials are identically equal to each other, that is, the coefficients of equal powers of x coincide. Finally, let us mention the relationship between the roots of an nth-degree equation and its coefficients. Let the polynomial

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n$$

have the roots $x_1, x_2, \dots, x_n$, so that there exists the factorization

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = (x - x_1)(x - x_2)\dots(x - x_n).$$

We then have the relations:

$$x_1 + x_2 + \dots + x_n = -p_1,$$

$$x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots + x_{n-1}x_n = p_2,$$

$$x_1x_2x_3 + \dots + x_{n-2}x_{n-1}x_n = -p_3,$$

$$\dots\dots\dots\dots\dots\dots$$

$$x_1x_2\dots x_n = \pm p_n.$$

1. Let x and y be two complex numbers.
Prove that

$$|x+y|^2+|x-y|^2=2\{|x|^2+|y|^2\}.$$

The symbol $|\alpha|$ denotes the modulus of the complex number α.

2. Find all the complex numbers satisfying the following condition

1° $\bar{x}=x^2$;

2° $\bar{x}=x^3$.

The symbol $\bar{x}$ denotes the number conjugate of x.

3. Prove that

$$\sqrt{(a_1+a_2+\ldots+a_n)^2+(b_1+b_2+\ldots+b_n)^2}\leqslant\sqrt{a_1^2+b_1^2}+\\ +\sqrt{a_2^2+b_2^2}+\ldots+\sqrt{a_n^2+b_n^2},$$

where a_i and b_i are any real numbers ($i=1, 2, 3, \ldots, n$).

4. Show that

$$(a+b+c)(a+b\varepsilon+c\varepsilon^2)(a+b\varepsilon^2+c\varepsilon)=\\ =a^3+b^3+c^3-3abc$$

if

$$\varepsilon^2+\varepsilon+1=0.$$

5. Prove that

$$(a^2+b^2+c^2-ab-ac-bc)\times\\ \times(x^2+y^2+z^2-xy-xz-yz)=\\ =X^2+Y^2+Z^2-XY-XZ-YZ$$

if

$$X=ax+cy+bz,$$
$$Y=cx+by+az,$$
$$Z=bx+ay+cz.$$

6. Given

$$x+y+z=A,$$
$$x+y\varepsilon+z\varepsilon^2=B,$$
$$x+y\varepsilon^2+z\varepsilon=C.$$

Here and in the next problem ε is determined by the equa lity

$$\varepsilon^2 + \varepsilon + 1 = 0.$$

1° Express x, y, z in terms of A, B, and C.
2° Prove that

$$|A|^2 + |B|^2 + |C|^2 = 3\{|x|^2 + |y|^2 + |z|^2\}.$$

7. Let

$$\begin{aligned} A &= x+y+z, & A' &= x'+y'+z', & AA' &= x''+y''+z'',\\ B &= x+y\varepsilon+z\varepsilon^2, & B' &= x'+y'\varepsilon+z'\varepsilon^2, & BB' &= x''+y''\varepsilon+z''\varepsilon^2;\\ C &= x+y\varepsilon^2+z\varepsilon, & C' &= x'+y'\varepsilon^2+z'\varepsilon, & CC' &= x''+y''\varepsilon^2+z''\varepsilon. \end{aligned}$$

Express x'', y'' and z'' in terms of x, y, z and x', y', z'.

8. Prove the identity

$$\begin{aligned} &(ax - by - cz - dt)^2 + (bx + ay - dz + ct)^2 +\\ &+ (cx + dy + az - bt)^2 + (dx - cy + bz + at)^2 =\\ &\qquad = (a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + t^2). \end{aligned}$$

9. Prove the following equalities

1° $\dfrac{\cos n\varphi}{\cos^n \varphi} = 1 - \binom{n}{2}\tan^2\varphi + \binom{n}{4}\tan^4\varphi - \ldots + A,$

where

$$A = (-1)^{\frac{n}{2}} \tan^n \varphi \quad \text{if } n \text{ is even,}$$

$$A = (-1)^{\frac{n-1}{2}} \binom{n}{n-1} \tan^{n-1}\varphi \quad \text{if } n \text{ is odd;}$$

2° $\dfrac{\sin n\varphi}{\cos^n \varphi} = \binom{n}{1}\tan\varphi - \binom{n}{3}\tan^3\varphi + \binom{n}{5}\tan^5\varphi + \ldots + A,$

where

$$A = (-1)^{\frac{n-2}{2}} \binom{n}{n-1} \tan^{n-1}\varphi \quad \text{if } n \text{ is even,}$$

$$A = (-1)^{\frac{n-1}{2}} \tan^n \varphi \quad \text{if } n \text{ is odd.}$$

Here and in the following problems

$$\binom{n}{k} = C_n^k = \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot 3\cdot\dots\cdot k}.$$

10. Prove the following equalities

1° $2^{2m}\cos^{2m} x = \sum_{k=0}^{k=m-1} 2\binom{2m}{k}\cos 2(m-k)x + \binom{2m}{m};$

2° $2^{2m}\sin^{2m} x = \sum_{k=0}^{k=m-1} (-1)^{m+k}\, 2\binom{2m}{k}\cos 2(m-k)x + \binom{2m}{m};$

3° $2^{2m}\cos^{2m+1} x = \sum_{k=0}^{k=m} \binom{2m+1}{k}\cos(2m-2k+1)x;$

4° $2^{2m}\sin^{2m+1} x = \sum_{k=0}^{k=m} (-1)^{m+k}\binom{2m+1}{k}\sin(2m-2k+1)x.$

11. Let

$$u_n = \cos\alpha + r\cos(\alpha+\theta) + r^2\cos(\alpha+2\theta) + \dots + r^n\cos(\alpha+n\theta),$$

$$v_n = \sin\alpha + r\sin(\alpha+\theta) + r^2\sin(\alpha+2\theta) + \dots + r^n\sin(\alpha+n\theta).$$

Show that

$$u_n = \frac{\cos\alpha - r\cos(\alpha-\theta) - r^{n+1}\cos[(n+1)\theta+\alpha] + r^{n+2}\cos(n\theta+\alpha)}{1-2r\cos\theta+r^2},$$

$$v_n = \frac{\sin\alpha \quad r\sin(\alpha-\theta) - r^{n+1}\sin[(n+1)\theta+\alpha] + r^{n+2}\sin(n\theta+\alpha)}{1-2r\cos\theta+r^2}.$$

12. Simplify the following sums

1° $S = 1 + n\cos\theta + \frac{n(n-1)}{1\cdot 2}\cos 2\theta + \dots = \sum_{k=0}^{k=n} C_n^k\cos k\theta, \quad (C_n^0 = 1);$

$$2° \quad S' = n \sin \theta + \frac{n(n-1)}{1 \cdot 2} \sin 2\theta + \ldots = \sum_{k=0}^{k=n} C_n^k \sin k\theta.$$

13. Prove the identity

$$\sin^{2p} \alpha + \sin^{2p} 2\alpha + \sin^{2p} 3\alpha + \ldots + \sin^{2p} n\alpha = \\ = \frac{1}{2} + n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2p-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2p}$$

if $\alpha = \frac{\pi}{2n}$ and $p < 2n$ (pa positive integer).

14. Prove that

1° The polynomial $x(x^{n-1} - na^{n-1}) + a^n(n-1)$ is divisible by $(x-a)^2$.

2° The polynomial $(1-x^n)(1+x) - 2nx^n(1-x) - n^2x^n(1-x)^2$ is divisible by $(1-x)^3$.

15. Prove that

1° $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y) \times (x^2+xy+y^2)$ if n is an odd number not divisible by 3.

2° $(x+y)^n - x^n - y^n$ is divisible by $xy(x+y) \times (x^2+xy+y^2)^2$ if n, when divided by 6, yields unity as a remainder, i.e. if $n \equiv 1 \pmod 6$.

16. Show that the following identities are true

1° $(x+y)^3 - x^3 - y^3 = 3xy(x+y)$;

2° $(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2+xy+y^2)$;

3° $(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)^2$.

17. Show that the expression

$$(x+y+z)^m - x^m - y^m - z^m \quad (m \text{ odd})$$

is divisible by

$$(x+y+z)^3 - x^3 - y^3 - z^3.$$

18. Find the condition necessary and sufficient for $x^3 + y^3 + z^3 + kxyz$ to be divisible by $x+y+z$.

19. Deduce the conditon at which $x^n - a^n$ is divisible by $x^p - a^p$ (n and p positive integers).

20. Find out whether the polynomial $x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$ (a, b, c, d positive integers) is divisible by

$$x^3 + x^2 + x + 1.$$

21. Find out at what n the polynomial $1 + x^2 + x^4 + \dots + x^{2n-2}$ is divisible by the polynomial $1 + x + x^2 + \dots + x^{n-1}$.

22. Prove that

1° The polynomial $(\cos\varphi + x\sin\varphi)^n - \cos n\varphi - x\sin n\varphi$ is divisible by $x^2 + 1$.

2° The polynomial $x^n \sin\varphi - \rho^{n-1} x \sin n\varphi + \rho^n \sin(n-1)\varphi$ is divisible by $x^2 - 2\rho x\cos\varphi + \rho^2$.

23. Find out at what values of p and q the binomial $x^4 + 1$ is divisible by $x^2 + px + q$.

24. Single out the real and imaginary parts in the expression $\sqrt{a + bi}$, i.e. represent this expression in the form $x + yi$, where x and y are real.

25. Find all the roots of the equation

$$x^n = 1.$$

26. Find the sum of the pth powers of the roots of the equation

$$x^n = 1 \ (p \text{ a positive integer}).$$

27. Let

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \quad (n \text{ a positive integer})$$

and let

$$A_k = x + y\varepsilon^k + z\varepsilon^{2k} + \dots + w\varepsilon^{(n-1)k} \qquad (k = 0, 1, 2, \dots, n-1),$$

where $x, y, z, \dots, u, w$ are n arbitrary complex numbers.

Prove that

$$\sum_{k=0}^{k=n-1} |A_k|^2 = n\{|x|^2 + |y|^2 + |z|^2 + \dots + |w|^2\}$$

(see Problem 6).

28. Prove the identities

$$1^\circ \ x^{2n} - 1 = (x^2 - 1) \sum_{k=1}^{k=n-1} \left(x^2 - 2x\cos\frac{k\pi}{n} + 1\right);$$

$$2^\circ \ x^{2n+1} - 1 = (x - 1) \prod_{k=1}^{k=n} \left(x^2 - 2x\cos\frac{2k\pi}{2n+1} + 1\right);$$

$$3^\circ\ x^{2n+1}-1=(x+1)\prod_{k=1}^{k=n}\left(x^2+2x\cos\frac{2k\pi}{2n+1}+1\right);$$

$$4^\circ\ x^{2n}+1=\prod_{k=0}^{k=n-1}\left(x^2-2x\cos\frac{(2k+1)\pi}{2n}+1\right).$$

29. Prove the identities

$$1^\circ\ \sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}};$$

$$2^\circ\ \cos\frac{2\pi}{2n+1}\cos\frac{4\pi}{2n+1}\ldots\cos\frac{2n\pi}{2n+1}=\frac{(-1)^{\frac{n}{2}}}{2^n}$$

if n is even.

30. Let the equation $x^n=1$ have the roots $1, \alpha, \beta, \gamma, \ldots, \lambda$. Show that

$$(1-\alpha)(1-\beta)(1-\gamma)\ldots(1-\lambda)=n.$$

31. Let

$$x_1, x_2, \ldots, x_n$$

be the roots of the equation

$$x^n+x^{n-1}+\ldots+x+1=0.$$

Compute the expression

$$\frac{1}{x_1-1}+\frac{1}{x_2-1}+\ldots+\frac{1}{x_n-1}.$$

32. Without solving the equations

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}+\frac{z^2}{\mu^2-c^2}=1,$$

$$\frac{x^2}{\nu^2}+\frac{y^2}{\nu^2-b^2}+\frac{z^2}{\nu^2-c^2}=1,$$

$$\frac{x^2}{\rho^2}+\frac{y^2}{\rho^2-b^2}+\frac{z^2}{\rho^2-c^2}=1,$$

find

$$x^2+y^2+z^2.$$

33. Prove that if $\cos\alpha + i\sin\alpha$ is the solution of the equation

$$x^n + p_1x^{n-1} + \dots + p_n = 0,$$

then $p_1\sin\alpha + p_2\sin 2\alpha + \dots + p_n\sin n\alpha = 0$ ($p_1, p_2, \dots, p_n$ are real).

34. If $a, b, c, \dots, k$ are the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0$$

($p_1, p_2, \dots, p_n$ are real), then prove that

$$(1+a^2)(1+b^2)\dots(1+k^2) = \\ = (1 - p_2 + p_4 - \dots)^2 + (p_1 - p_3 + p_5 - \dots)^2.$$

35. Show that if the equations

$$x^3 + px + q = 0$$
$$x^3 + p'x + q' = 0$$

have a common root, then

$$(pq' - qp')(p - p')^2 = (q - q')^3.$$

36. Prove the following identities

1° $$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \\ = \sqrt[3]{\frac{1}{2}(5-3\sqrt[3]{7})};$$

2° $$\sqrt[3]{\cos\frac{2\pi}{9}} + \sqrt[3]{\cos\frac{4\pi}{9}} + \sqrt[3]{\cos\frac{8\pi}{9}} = \sqrt[3]{\frac{1}{2}(3\sqrt[3]{9}-6)}.$$

37. Let $a+b+c=0$.
Put

$$a^k + b^k + c^k = s_k.$$

Prove the following relations (see Problems 23, 24, 26 of Sec. 1)

$$2s_4 = s_2^2, \qquad 6s_5 = 5s_2s_3,$$
$$6s_7 = 7s_3s_4, \qquad 10s_7 = 7s_2s_5,$$
$$25s_7s_3 = 21s_5^2, \qquad 50s_7^2 = 49s_4s_5^2,$$
$$s_{n+3} = abcs_n + \frac{1}{2}s_2s_{n+1}.$$

38. 1° Given

$$x + y = u + v,$$
$$x^2 + y^2 = u^2 + v^2.$$

Prove that

$$x^n + y^n = u^n + v^n$$

for any n.

2° Given

$$x + y + z = u + v + t,$$
$$x^2 + y^2 + z^2 = u^2 + v^2 + t^2,$$
$$x^3 + y^3 + z^3 = u^3 + v^3 + t^3.$$

Prove that

$$x^n + y^n + z^n = u^n + v^n + t^n$$

for any n.

39. Let

$$A = x_1 + x_2\varepsilon + x_3\varepsilon^2, \quad B = x_1 + x_2\varepsilon^2 + x_3\varepsilon,$$

where

$$\varepsilon^2 + \varepsilon + 1 = 0,$$

and x_1, x_2, x_3 are the roots of the cubic equation

$$x^3 + px + q = 0.$$

Prove that A^3 and B^3 are the roots of the quadratic equation

$$z^2 + 27qz - 27p^3 = 0.$$

40. Solve the equation

$$(x + a)(x + b)(x + c)(x + d) = m$$

if

$$a + b = c + d.$$

41. Solve the equation

$$(x + a)^4 + (x + b)^4 = c.$$

42. Solve the equation

$$(x + b + c)(x + a + c)(x + a + b)(a + b + c) - abcx = 0.$$

43. Solve the equation

$$x^3 + 3ax^2 + 3(a^2 - bc)x + a^3 + b^3 + c^3 - 3abc = 0.$$

44. Solve the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

if

$$a + b = b + c + d = d + e.$$

45. Solve the equation

$$(a + b + x)^3 - 4(a^3 + b^3 + x^3) - 12abx = 0.$$

46. Solve the equation

$$x^2 + \frac{a^2x^2}{(a+x)^2} = m \quad (a \text{ and } m > 0).$$

Deduce the condition under which all the roots are real, and determine the number of positive and negative roots.

47. Solve the equation

$$\frac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+1)(a^4+10a^2+5)} = ax.$$

48. Solve the equation

$$1 + \frac{a_1}{x-a_1} + \frac{a_2x}{(x-a_1)(x-a_2)} + \frac{a_3x^2}{(x-a_1)(x-a_2)(x-a_3)} + \ldots +$$
$$+ \frac{a_{2m}x^{2m-1}}{(x-a_1)(x-a_2)\ldots(x-a_{2m})} = \frac{2px^m - p^2}{(x-a_1)(x-a_2)\ldots(x-a_{2m})}.$$

49. 1° Solve the equation

$$x^3 + px^2 + qx + r = 0$$

if $x_1^2 = x_2x_3$.

2° Solve the equation

$$x^3 + px^2 + qx + r = 0 \quad \text{if } x_1 = x_2 + x_3.$$

50. 1° Solve the system

$$y^3 + z^3 + a^3 = 3ayz$$
$$z^3 + x^3 + b^3 = 3bzx$$
$$x^3 + y^3 + c^3 = 3cxy.$$

2° Solve the system

$$x^4 - a = y^4 - b = z^4 - c = u^4 - d = xyzu$$

if $a + b + c + d = 0$.

51. In the expansion $1 + (1 + x) + \ldots + (1 + x)^n$ in powers of x find the term containing x^k.

52. Prove that the coefficient of x^s in the expansion in powers of x of the expression $\{(s - 2) x^2 + nx - s\} (x+1)^n$ is equal to

$$nC_n^{s-2}.$$

53. Prove that for $x > 1$ $px^q - qx^p - p + q > 0$ (p, q positive integers and $q > p$).

54. Let x and a be positive numbers. Determine the greatest term in the expansion of $(x + a)^n$.

55. Prove that

1° $$i^m - i(i-1)^m + \frac{i(i-1)}{1\cdot 2}(i-2)^m + \ldots + (-1)^{i-1} i \cdot 1^m = 0$$

if $i > m$.

2° $$m^m - m(m-1)^m + \frac{m(m-1)}{1\cdot 2}(m-2)^m + \ldots + (-1)^{m-1} m = m!$$

(i and m positive integers).

56. Prove the identity

$$(x^2 + a^2)^n = \{x^n - C_n^2 x^{n-2} a^2 + C_n^4 x^{n-4} a^4 - \ldots\}^2 + \{C_n^1 x^{n-1} a - C_n^3 x^{n-3} a^3 + \ldots\}^2.$$

57. Determine the coefficient of x^l ($l = 0, 1, \ldots, 2n$) in the following products

1° $\{1 + x + x^2 + \ldots + x^n\}\{1 + x + x^2 + \ldots + x^n\}$;

2° $\{1 + x + x^2 + \ldots + x^n\}\{1 - x + x^2 - x^3 + \ldots + (-1)^n x^n\}$;

3° $\{1 + 2x + 3x^2 + \ldots + (n+1) x^n\}\{1 + 2x + 3x^2 + \ldots + (n+1) x^n\}$;

4° $\{1 + 2x + 3x^2 + \ldots + (n+1) x^n\}\{1 - 2x + 3x^2 - \ldots + (-1)^n (n+1) x^n\}$.

58. Prove that

$1^\circ\ 1+C_n^2+C_n^4+\ldots=C_n^1+C_n^3+\ldots=2^{n-1};$

$2^\circ\ C_{2n}^1+C_{2n}^3+\ldots+C_{2n}^{n-1}=2^{2n-2}$ if n is even;

$3^\circ\ 1+C_{2n}^2+\ldots+C_{2n}^{n-1}=2^{2n-2}$ if n is odd.

59. Prove the identities

$$1^\circ\ C_n^0+C_n^3+C_n^6+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right);$$

$$2^\circ\ C_n^1+C_n^4+C_n^7+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right);$$

$$3^\circ\ C_n^2+C_n^5+C_n^8+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{(n-4)\pi}{3}\right).$$

60. Prove that

$$1^\circ\ C_n^0+C_n^4+C_n^8+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right);$$

$$2^\circ\ C_n^1+C_n^5+C_n^9+\ldots=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right);$$

$$3^\circ\ C_n^2+C_n^6+C_n^{10}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right);$$

$$4^\circ\ C_n^3+C_n^7+C_n^{11}+\ldots=\frac{1}{2}\left(2^{n-1}-2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right).$$

61. Prove the equality

$$1^2+2^2+\ldots+n^2=C_{n+1}^2+2\left(C_n^2+C_{n-1}^2+\ldots+C_2^2\right).$$

62. If a_1, a_2, a_3 and a_4 are four successive coefficients in the expansion of $(1+x)^n$ in powers of x, then

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$

63. Prove the identity

$$\frac{1}{1!\,(n-1)!}+\frac{1}{3!\,(n-3)!}+\frac{1}{5!\,(n-5)!}+\ldots+\frac{1}{(n-1)!\,1!}=\frac{2^{n-1}}{n!}$$

(n even).

64. Find the magnitude of the sum

$$s = C_n^1 - 3C_n^3 + 3^2C_n^5 - 3^3C_n^7 + \dots .$$

65. Find the magnitudes of the following sums

$$\sigma = 1 - C_n^2 + C_n^4 - C_n^6 + \dots,$$

$$\sigma' = C_n^1 - C_n^3 + C_n^5 - C_n^7 + \dots .$$

66. Prove the identities

1° $C_n^0 + 2C_n^1 + 3C_n^2 + 4C_n^3 + \dots + (n+1)C_n^n = (n+2)2^{n-1}$;

2° $C_n^1 - 2C_n^2 + 3C_n^3 + \dots + (-1)^{n-1}nC_n^n = 0.$

67. Prove that

$$\frac{1}{2}C_n^1 - \frac{1}{3}C_n^2 + \frac{1}{4}C_n^3 + \dots + \frac{(-1)^{n-1}}{n+1}C_n^n = \frac{n}{n+1}.$$

68. Prove that

1° $1 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + \dots + \frac{1}{n+1}C_n^n = \frac{2^{n+1}-1}{n+1}$;

2° $2C_n^0 + \frac{2^2C_n^1}{2} + \frac{2^3C_n^2}{3} + \frac{2^4C_n^3}{4} + \dots + \frac{2^{n+1}C_n^n}{n+1} = \frac{3^{n+1}-1}{n+1}.$

69. Prove the identity

$$C_n^1 - \frac{1}{2}C_n^2 + \frac{1}{3}C_n^3 + \dots + \frac{(-1)^{n-1}}{n}C_n^n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

70. Prove that

1° $C_n^n + C_{n+1}^n + C_{n+2}^n + \dots + C_{n+k}^n = C_{n+k+1}^{n+1}$;

2° $C_n^0 - C_n^1 + C_n^2 + \dots + (-1)^h C_n^h = (-1)^h C_{n-1}^h$.

71. Show that the following equalities exist

1° $C_n^0C_m^p + C_n^1C_m^{p-1} + \dots + C_n^pC_m^0 = C_{m+n}^p$;

2° $C_n^0C_n^r + C_n^1C_n^{r+1} + \dots + C_n^{n-r}C_n^n = \frac{2n!}{(n-r)!\,(n+r)!}.$

72. Prove the following identities

1° $(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 + \dots + (C_n^n)^2 = C_{2n}^n$;

2° $(C_{2n}^0)^2 - (C_{2n}^1)^2 + (C_{2n}^2)^2 - \dots + (C_{2n}^{2n})^2 = (-1)^n C_{2n}^n$;

3° $(C_{2n+1}^{0})^2-(C_{2n+1}^{1})^2+(C_{2n+1}^{2})^2-\ldots-(C_{2n+1}^{2n+1})^2=0;$

4° $(C_n^1)^2+2(C_n^2)^2+\ldots+n(C_n^n)^2=\frac{(2n-1)!}{(n-1)!\,(n-1)!}.$

73. Let $f(x)$ be a polynomial leaving the remainder A when divided by $x-a$ and the remainder B when divided by $x-b$ $(a\neq b)$. Find the remainder left by this polynomial when divided by $(x-a)(x-b)$.

74. Let $f(x)$ be a polynomial leaving the remainder A when divided by $x-a$, the remainder B when divided by $x-b$ and the remainder C when divided by $x-c$. Find the remainder left by this polynomial when divided by $(x-a)(x-b)(x-c)$ if a, b and c are not equal to one another.

75. Find the polynomial in x of degree $(m-1)$ which at m different values of x, $x_1, x_2, \ldots, x_m$, attains respectively the values $y_1, y_2, \ldots, y_m$.

76. Let $f(x)$ be a polynomial leaving the remainder A_1 when divided by $x-a_1$, the remainder A_2 when divided by $x-a_2, \ldots,$ and, finally, the remainder A_m when divided by $x-a_m$. Find the remainder left by the polynomial, when divided by $(x-a_1)(x-a_2)\ldots(x-a_m)$.

77. Prove that if $x_1, x_2, \ldots, x_m$ are m different arbitrary quantities, $f(x)$ is a polynomial of degree less than m, then there exists the identity

$$f(x)=f(x_1)\frac{(x-x_2)(x-x_3)\ldots(x-x_m)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)}+$$
$$+f(x_2)\frac{(x-x_1)(x-x_3)\ldots(x-x_m)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)}+\ldots+$$
$$+f(x_m)\frac{(x-x_1)(x-x_2)\ldots(x-x_{m-1})}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})}.$$

78. Prove that if $f(x)$ is a polynomial whose degree is less than, or equal to, $m-2$ and $x_1, x_2, \ldots, x_m$ are m arbitrary unequal quantities, then there exists the identity

$$\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)}+\frac{f(x_2)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)}+$$
$$+\ldots+\frac{f(x_m)}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})}=0.$$

79. Put

$$s_n = \frac{x_1^n}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_m)} + \frac{x_2^n}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_m)} + \dots + \frac{x_m^n}{(x_m-x_1)(x_m-x_2)\dots(x_m-x_{m-1})}$$

($x_1, x_2, \dots, x_m$ are m arbitrary unequal quantities). Show that $s_n = 0$ if $0 \leqslant n < m-1$, and $s_{m-1} = 1$, and compute s_n if $n \geqslant m$.

80. Compute the following

$$s_{-n} = \frac{x_1^{-n}}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_m)} + \frac{x_2^{-n}}{(x_2-x_1)(x_2-x_3)\dots(x_2+x_m)} + \dots + \frac{x_m^{-n}}{(x_m-x_1)(x_m-x_2)\dots(x_m-x_{m-1})} \quad (n = 1, 2, 3, \dots).$$

81. Show that if $f(x)$ is a polynomial whose degree is less than m, then the fraction

$$\frac{f(x)}{(x-x_1)(x-x_2)\dots(x-x_m)}$$

($x_1, x_2, \dots, x_m$ are arbitrary quantities not equal to each other) can be represented as a sum of m partial fractions

$$\frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \dots + \frac{A_m}{x-x_m},$$

where $A_1, A_2, \dots, A_m$ are independent of x.

82. Solve the system of equations

$$\frac{x_1}{a_1-b_1} + \frac{x_2}{a_1-b_2} + \dots + \frac{x_n}{a_1-b_n} = 1$$

$$\frac{x_1}{a_2-b_1} + \frac{x_2}{a_2-b_2} + \dots + \frac{x_n}{a_2-b_n} = 1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{x_1}{a_n-b_1} + \frac{x_2}{a_n-b_2} + \dots + \frac{x_n}{a_n-b_n} = 1.$$

83. Prove that the following identity is true

$$\frac{n!}{(x+1)(x+2)\dots(x+n)} = \frac{C_n^1}{x+1} - \frac{2C_n^2}{x+2} + \frac{3C_n^3}{x+3} - \dots + (-1)^{n+1}\frac{nC_n^n}{x+n}.$$

In particular,

$$\frac{1}{n+1}=\frac{C_n^1}{2}-\frac{2}{3}C_n^2+\frac{3}{4}C_n^3-\frac{4}{5}C_n^4+\dots .$$

84. Prove the identity

$$(-1)^n\frac{a_1a_2\dots a_n}{b_1b_2\dots b_n}+\frac{(a_1-b_1)(a_2-b_1)\dots(a_n-b_1)}{b_1(b_1-b_2)\dots(b_1-b_n)}+$$
$$+\frac{(a_1-b_2)(a_2-b_2)\dots(a_n-b_2)}{b_2(b_2-b_1)\dots(b_2-b_n)}+\dots+$$
$$+\frac{(a_1-b_n)\dots(a_n-b_n)}{b_n(b_n-b_1)\dots(b_n-b_{n-1})}=(-1)^n.$$

85. Prove the identity

$$\frac{(x+\beta)\dots(x+n\beta)}{(x-\beta)\dots(x-n\beta)}-1=$$
$$=\sum_{r=1}^{r=n}(-1)^{n-r}\frac{n(n+r)(n^2-1^2)(n^2-2^2)\dots[n^2-(r-1)^2]}{(r!)^2}\cdot\frac{r\beta}{x-r\beta}.$$

86. Given a series of numbers $c_0, c_1, c_2, \dots, c_k, c_{k+1}, \dots$. Put $\Delta c_k = c_{k+1} - c_k$, so that using the given series we can form a new one

$$\Delta c_0,\ \Delta c_1,\ \Delta c_2,\ \dots .$$

We then put

$$\Delta^2 c_k = \Delta c_{k+1} - \Delta c_k$$

so as to get one more series: $\Delta^2 c_0$, $\Delta^2 c_1$, $\Delta^2 c_2$, ... and so forth.

Prove the following formulas

1° $$c_{k+n}=c_k+\frac{n}{1}\Delta c_k+\frac{n(n-1)}{1\cdot 2}\Delta^2 c_k+$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\Delta^3 c_k+\dots+\Delta^n c_k;$$

2° $$\Delta^n c_k=c_{k+n}-\frac{n}{1}c_{k+n-1}+\frac{n(n-1)}{1\cdot 2}c_{k+n-2}+\dots+(-1)^n c_k.$$

87. Show that if $f(x)$ is any polynomial of nth degree in x, then there exists the following identity

$$f(x)=f(0)+\frac{x}{1}\Delta f(0)+\frac{x(x-1)}{1\cdot 2}\Delta^2 f(0)+\dots+\frac{x(x-1)\dots(x-n+1)}{n!}\Delta^n f(0),$$

where $\Delta f(0)$, $\Delta^2 f(0)$, ..., $\Delta^n f(0)$ are obtained, proceeding from the basic series: $f(0)$, $f(1)$, $f(2)$, ...

88. Show that if

$$x^n=A_0+\frac{A_1}{1}(x-1)+\frac{A_2}{2!}(x-1)(x-2)+\dots+\frac{A_n}{n!}(x-1)(x-2)\dots(x-n),$$

then $A_s=(s+1)^n-C_s^1 s^n+C_s^2(s-1)^n+\dots+(-1)^s C_s^s\cdot 1^n$.

89. Prove the identity

$$\frac{n!}{x(x+1)\dots(x+n)}\left\{\frac{1}{x}+\frac{1}{x+1}+\dots+\frac{1}{x+n}\right\}=\frac{1}{x^2}-\frac{C_n^1}{(x+1)^2}+\frac{C_n^2}{(x+2)^2}+\dots+(-1)^n\frac{1}{(x+n)^2}.$$

90. Let

$$\varphi_k(x)=x(x-1)(x-2)\dots(x-k+1).$$

Prove that the following identity exists

$$\varphi_n(x+y)=\varphi_n(x)+C_n^1\varphi_{n-1}(x)\varphi_1(y)+C_n^2\varphi_{n-2}(x)\varphi_2(y)+\dots+C_n^{n-1}\varphi_1(x)\varphi_{n-1}(y)+\varphi_n(y).$$

91. Prove the following identities

1° $$x^n+y^n=p^n-\frac{n}{1}p^{n-2}q+\frac{n(n-3)}{1\cdot 2}p^{n-4}q^2-\dots+(-1)^r\frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!}p^{n-2r}q^r+\dots;$$

2° $$\frac{x^{n+1}-y^{n+1}}{x-y}=p^n-C_{n-1}^1p^{n-2}q+C_{n-2}^2p^{n-4}q^2-\dots+(-1)^rC_{n-r}^rp^{n-2r}q^r+\dots,$$

where

$$p=x+y, \quad q=xy.$$

92. Let $x+y=1$.

Prove that

$$x^m(1+C_m^1 y+C_{m+1}^2 y^2+\dots+C_{2m-2}^{m-1}y^{m-1})+ \\ +y^m(1+C_m^1 x+\dots+C_{2m-2}^{m-1}x^{m-1})=1.$$

93. Prove that the following identity is true

$$\frac{1}{(x-a)^m(x-b)^m}=\frac{1}{(a-b)^m}\left\{\frac{1}{(x-a)^m}+\frac{C_m^1}{(x-a)^{m-1}(b-a)}+\right. \\ \left.+\frac{C_{m+1}^2}{(x-a)^{m-2}(b-a)^2}+\dots+\frac{C_{2m-2}^{m-1}}{(x-a)(b-a)^{m-1}}\right\}+ \\ +\frac{1}{(b-a)^m}\left\{\frac{1}{(x-b)^m}+\frac{C_m^1}{(x-b)^{m-1}(a-b)}+\dots+\right. \\ \left.+\frac{C_{2m-2}^{m-1}}{(x-b)(a-b)^{m-1}}\right\}.$$

94. Show that constants A_1, A_2, A_3 can always be chosen so that the following identity takes place

$$(x+y)^n=x^n+y^n+A_1xy(x^{n-2}+y^{n-2})+ \\ +A_2x^2y^2(x^{n-4}+y^{n-4})+\dots .$$

Determine these constants.

95. Solve the system

$$\begin{aligned} x_1+x_2 &= a_1 \\ x_1y_1+x_2y_2 &= a_2 \\ x_1y_1^2+x_2y_2^2 &= a_3 \\ x_1y_1^3+x_2y_2^3 &= a_4. \end{aligned}$$

Show how the general system is solved

$$\begin{aligned} x_1+x_2+x_3+\dots+x_{n-1}+x_n &= a_1 && (1) \\ x_1y_1+x_2y_2+\dots+x_ny_n &= a_2 && (2) \\ x_1y_1^2+x_2y_2^2+\dots+x_ny_n^2 &= a_3 && (3) \\ \dots\dots\dots\dots\dots\dots\dots \\ x_1y_1^{2n-1}+x_2y_2^{2n-1}+\dots+x_ny_n^{2n-1} &= a_{2n}. && (2n) \end{aligned}$$

96. Solve the system

$$\begin{aligned} x+y+z+u+v&=2\\ px+qy+rz+su+tv&=3\\ p^2x+q^2y+r^2z+s^2u+t^2v&=16\\ p^3x+q^3y+r^3z+s^3u+t^3v&=31\\ p^4x+q^4y+r^4z+s^4u+t^4v&=103\\ p^5x+q^5y+r^5z+s^5u+t^5v&=235\\ p^6x+q^6y+r^6z+s^6u+t^6v&=674\\ p^7x+q^7y+r^7z+s^7u+t^7v&=1\,669\\ p^8x+q^8y+r^8z+s^8u+t^8v&=4\,526\\ p^9x+q^9y+r^9z+s^9u+t^9v&=11\,595. \end{aligned}$$

97. Let m and μ be positive integers ($\mu\leqslant m$). Put

$$\frac{(1-x^m)(1-x^{m-1})\dots(1-x^{m-\mu+1})}{(1-x)(1-x^2)\dots(1-x^{\mu})}=(m,\ \mu).$$

Prove that

1° $(m,\ \mu)=(m,\ m-\mu)$;

2° $(m,\ \mu+1)=(m-1,\ \mu+1)+x^{n-\mu-1}(m-1,\ \mu)$;

3° $(m,\ \mu+1)=(\mu,\ \mu)+x(\mu+1,\ \mu)+x^2(\mu+2,\ \mu)+\dots+$
$+x^{m-\mu-1}(m-1,\ \mu)$;

4° $(m,\ \mu)$ is a polynomial in x;

5° $1-(m,\ 1)+(m,\ 2)-(m,\ 3)+\dots$ is equal to

$(1-x)(1-x^2)\dots(1-x^{m-1})$ if m is even,

0 if m is odd.

(Gauss, *Summatio quarumdam serierum singularium*, Werke, Bd. II).

98. Prove that

1° $(1+xz)(1+x^2z)\dots(1+x^nz)=$

$$=1+\sum_{k=1}^{k=n}\frac{(1-x^n)(1-x^{n-1})\dots(1-x^{n-k+1})}{(1-x^1)(1-x^2)\dots(1-x^k)}x^{\frac{k(k+1)}{2}}z^k;$$

2° $(1+xz)(1+x^3z)\dots(1+x^{2n-1}z)=$

$$=1+\sum_{k=1}^{k=n}\frac{(1-x^{2n})(1-x^{2n-2})\dots(1-x^{2n-2k+2})}{(1-x^2)(1-x^4)\dots(1-x^{2k})}x^{k^2}z^k.$$

99. Let

$$p_k=(1-x)(1-x^2)\dots(1-x^k).$$

Prove that

$$\frac{1}{p_n}-\frac{x}{p_1p_{n-1}}+\frac{x^3}{p_2p_{n-2}}-\dots\pm\frac{x^{\frac{n(n+1)}{1\cdot 2}}}{p_n}=1.$$

100. Determine the coefficients $C_0, C_1, C_2, \dots, C_n$ in the following identity

$$(1+xz)(1+xz^{-1})(1+x^3z)(1+x^3z^{-1})\dots\times$$
$$\times(1+x^{2n-1}z)(1+x^{2n-1}z^{-1})=C_0+C_1(z+z^{-1})+$$
$$+C_2(z^2+z^{-2})+\dots+C_n(z^n+z^{-n}).$$

101. Let

$$u_k=\frac{\sin 2nx\sin(2n-1)x\dots\sin(2n-k+1)x}{\sin x\sin 2x\dots\sin kx}.$$

Prove that

1° $1-u_1+u_2-u_3+\dots+u_{2n}=$

$$=2^n\cdot(1-\cos x)(1-\cos 3x)\dots[1-\cos(2n-1)x];$$

2° $1-u_1^2+u_2^2-u_3^2+\dots+u_{2n}^2=$

$$=(-1)^n\frac{\sin(2n+2)x\sin(2n+4)x\dots\sin 4nx}{\sin 2x\sin 4x\dots\sin 2nx}.$$

7. PROGRESSIONS AND SUMS

Solution of problems regarding the arithmetic and geometric progressions treated in the present section requires only knowledge of elementary algebra. As far as the summing of finite series is concerned, it is performed using the method of finite differences. Let it be required to find the sum

$f(1)+f(2)+\ldots+f(n)$. Find the function $F(k)$ which would satisfy the relationship

$$F(k+1)-F(k)=f(k).$$

Then it is obvious that

$$f(1)+f(2)+\ldots+f(n)=[F(2)-F(1)]+ \\ +[F(3)-F(2)]+\ldots+[F(n+1)-F(n)]= \\ =F(n+1)-F(1).$$

1. Let a^2, b^2, c^2 form an arithmetic progression. Prove that the quantities $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ also form an arithmetic progression.

2. Prove that if a, b and c are respectively the pth, qth and rth terms of an arithmetic progression, then

$$(q-r)a+(r-p)b+(p-q)c=0.$$

3. Let in an arithmetic progression $a_p=q$; $a_q=p$ (a_n is the nth term of the progression). Find a_m.

4. In an arithmetic progression $S_p=q$; $S_q=p$ (S_n is the sum of the first n terms of the progression). Find S_{p+q}.

5. Let in an arithmetic progression $S_p=S_q$. Prove that $S_{p+q}=0$.

6. Given in an arithmetic progression $\frac{S_m}{S_n}=\frac{m^2}{n^2}$. Prove that $\frac{a_m}{a_n}=\frac{2m-1}{2n-1}$.

7. Show that any power n^k ($k\geqslant 2$ an integer) can be represented in the form of a sum of n successive odd numbers.

8. Let the sequence $a_1, a_2, \ldots, a_n$ form an arithmetic progression and $a_1=0$. Simplify the expression

$$S=\frac{a_3}{a_2}+\frac{a_4}{a_3}+\ldots+\frac{a_n}{a_{n-1}}-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right).$$

9. Prove that in any arithmetic progression

$$a_1, a_2, a_3, \ldots$$

we have

$$S=\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+$$
$$+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

10. Show that in any arithmetic progression

$$a_1, a_2, a_3, \ldots$$

we have

$$S=a_1^2-a_2^2+a_3^2-a_4^2+\ldots+a_{2k-1}^2-a_{2k}^2=\frac{k}{2k-1}(a_1^2-a_{2k}^2).$$

11. Let $S(n)$ be the sum of the first n terms of an arithmetic progression.

Prove that

1° $S(n+3)-3S(n+2)+3S(n+1)-S_{(n)}=0.$

2° $S(3n)=3\{S(2n)-S(n)\}.$

12. Let the sequence $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$ be an arithmetic progression.

Prove that the sequence $S_1, S_2, S_3, \ldots$, where

$S_1=a_1+a_2+\ldots+a_n,$

$S_2=a_{n+1}+\ldots+a_{2n},\quad S_3=a_{2n+1}+\ldots+a_{3n},\ldots,$

is an arithmetic progression as well whose common difference is n^2 times greater than the common difference of the given progression.

13. Prove that if a, b, c are respectively the pth, qth and rth terms both of an arithmetic and a geometric progressions simultaneously, then

$$a^{b-c}\cdot b^{c-a}\cdot c^{a-b}=1.$$

14. Prove that

$$(1+x+x^2+\ldots+x^n)^2-x^n=$$
$$=(1+x+x^2+\ldots+x^{n-1})(1+x+x^2+\ldots+x^{n+1}).$$

15. Let S_n be the sum of the first n terms of a geometric progression.

Prove that $S_n(S_{3n}-S_{2n})=(S_{2n}-S_n)^2$.

16. Let the numbers $a_1, a_2, a_3, \ldots$ form a geometric progression.

Knowing the sums

$$S=a_1+a_2+a_3+\dots+a_n,\quad S'=\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n},$$

find the product $P=a_1a_2\dots a_n$.

17. If $a_1, a_2, \dots, a_n$ are real, then the equality

$$(a_1^2+a_2^2+\dots+a_{n-1}^2)(a_2^2+a_3^2+\dots+a_n^2)=$$
$$=(a_1a_2+a_2a_3+\dots+a_{n-1}a_n)^2$$

is possible if and only if $a_1, a_2, \dots, a_n$ form a geometric progression. Prove this.

18. Let $a_1, a_2, \dots, a_n$ be a geometric progression with ratio q and let $S_m=a_1+\dots+a_m$.

Find simpler expressions for the following sums

1° $S_1+S_2+\dots+S_n$;

2° $\dfrac{1}{a_1^2-a_2^2}+\dfrac{1}{a_2^2-a_3^2}+\dots+\dfrac{1}{a_{n-1}^2-a_n^2}$;

3° $\dfrac{1}{a_1^k+a_2^k}+\dfrac{1}{a_2^k+a_3^k}+\dots+\dfrac{1}{a_{n-1}^k+a_n^k}$.

19. Prove that in any arithmetic progression, whose common difference is not equal to zero, the product of two terms equidistant from the extreme terms is the greater the closer these terms are to the middle term.

20. An arithmetic and a geometric progression with positive terms have the same number of terms and equal extreme terms. For which of them is the sum of terms greater?

21. The first two terms of an arithmetic and a geometric progression with positive terms are equal. Prove that all other terms of the arithmetic progression are not greater than the corresponding terms of the geometric progression.

22. Find the sum of n terms of the series

$$S_n=1\cdot x+2x^2+3x^3+\dots+nx^n.$$

23. Let $a_1, a_2, \dots, a_n$ form an arithmetic progression and $u_1, u_2, \dots, u_n$ a geometric one. Find the expression for the sum

$$s=a_1u_1+a_2u_2+\dots+a_nu_n.$$

24. Find the sum

$$\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\ldots+\left(x^n+\frac{1}{x^n}\right)^2$$

25. Let

$$S_k=1^k+2^k+3^k+\ldots+n^k.$$

Prove that

$$S_1=\frac{n(n+1)}{1\cdot 2},\quad S_2=\frac{n(n+1)(2n+1)}{6},\quad S_3=\frac{n^2(n+1)^2}{4}$$

26. Prove the following general formula

$$(k+1)S_k+\frac{(k+1)k}{1\cdot 2}S_{k-1}+\frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}S_{k-2}+\ldots+$$
$$+(k+1)S_1+S_0=(n+1)^{k+1}-1.$$

27. Put

$$1^k+2^k+\ldots+n^k=S_k(n).$$

Prove the formula

$$nS_k(n)=S_{k+1}(n)+S_k(n-1)+S_k(n-2)+\ldots+$$
$$+S_k(2)+S_k(1).$$

28. 1° Prove that

$$1^k+2^k+3^k+\ldots+n^k=An^{k+1}+Bn^k+Cn^{k-1}+\ldots+Ln,$$

i.e. that the sum $S_k(n)$ can be represented as a polynomial of the $(k+1)$th degree in n with coefficients independent of n and without a constant term.

2° Show that $A=\frac{1}{k+1}$, and $B=\frac{1}{2}$.

29. Show that the following formulas take place

$$S_4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30},$$

$$S_5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12},$$

$$S_6=\frac{6n^7+21n^6+21n^5-7n^3+n}{42}=$$
$$=\frac{n(n+1)(2n+1)[3n^2(n+1)^2-(3n^2+3n-1)]}{42},$$

$$S_7 = \frac{3n^8 + 12n^7 + 14n^6 - 7n^4 + 2n^2}{24} =$$

$$= \frac{n^2(n+1)^2[3n^2(n+1)^2 - 2(2n^2+2n-1)]}{24}.$$

30. Prove that the following relations take place

$$S_3 = S_1^2, \quad 4S_1^3 = S_3 + 3S_5, \quad 2S_5 + S_3 = 3S_2^2, \quad S_5 + S_7 = 2S_3^2.$$

31. Consider the numbers B_0, B_1, B_2, B_3, B_4, . . . determined by the symbolic equality

$$(B+1)^{k+1} - B^{k+1} = k+1 \qquad (k = 0, 1, 2, 3, \ldots)$$

and the initial value $B_0 = 1$. Expanding the left member of this equality according to the binomial formula, we have to replace the exponents by subscripts everywhere. Thus, the above symbolic equality is identical to the following common equality

$$B_{k+1} + C_{k+1}^1 B_k + C_{k+1}^2 B_{k-1} + \ldots + C_{k+1}^k B_1 + B_0 - B_{k+1} = k+1.$$

1° Compute B_0, B_1, B_2, ..., B_{10} with the aid of this equality.

2° Show that the following formula takes place

$$1^k + 2^k + 3^k + \ldots + n^k =$$

$$= \frac{1}{k+1}\{n^{k+1} + C_{k+1}^1 B_1 n^k + C_{k+1}^2 B_2 n^{k-1} + \ldots + C_{k+1}^k B_k n\}.$$

32. Let x_1, x_2, ..., x_n form an arithmetic progression. It is known that

$$x_1 + x_2 + \ldots + x_n = a, \quad x_1^2 + x_2^2 + \ldots + x_n^2 = b^2.$$

Determine this progression.

33. Determine the sums of the following series

1° $1 + 4x + 9x^2 + \ldots + n^2 x^{n-1}$;

2° $1^3 + 2^3 x + 3^3 x^2 + \ldots + n^3 x^{n-1}$.

34. Determine the sums of the following series

1° $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots + \frac{2n-1}{2^{n-1}}$;

2° $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \ldots + (-1)^{n-1}\frac{2n-1}{2^{n-1}}$.

35. Determine the sums of the following series

1° $1-2+3-4+\ldots+(-1)^{n-1}n$;

2° $1^2-2^2+3^2-\ldots+(-1)^{n-1}n^2$;

3° $1-3^2+5^2-7^2+\ldots-(4n-1)^2$;

4° $2\cdot 1^2+3\cdot 2^2+\ldots+(n+1)n^2$.

36. Find the sum of n numbers of the form 1, 11, 111, 1111, ...

37. Prove the identity

$$x^{4n+2}+y^{4n+2}=$$
$$=\{x^{2n+1}-2x^{2n-1}y^2+2x^{2n-3}y^4-\ldots+(-1)^n 2xy^{2n}\}^2+$$
$$=\{y^{2n+1}-2y^{2n-1}x^2+2y^{2n-3}x^4-\ldots+(-1)^n 2yx^{2n}\}^2.$$

38. Find the sum of products of the numbers 1, a, a^2, ..., a^{n-1}, taken pairwise.

39. Prove the identity

$$\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+2\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+\ldots+(n-1)\left(x+\frac{1}{x}\right)+n=$$
$$=\frac{1}{x^{n-1}}\left(\frac{x^n-1}{x-1}\right)^2.$$

40. Prove the identity

1° $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=1-\frac{1}{n+1};$$

2° $$\frac{1}{1\cdot 2\cdot 3}+\frac{2}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=$$
$$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right);$$

3° $$\frac{1}{1\cdot 3\cdot 5}+\frac{2}{3\cdot 5\cdot 7}+\ldots+\frac{n}{(2n-1)(2n+1)(2n+3)}=$$
$$=\frac{n(n+1)}{2(2n+1)(2n+3)}.$$

41. Compute the sum

$$S=\frac{1^4}{1\cdot 3}+\frac{2^4}{3\cdot 5}+\frac{3^4}{5\cdot 7}+\ldots+\frac{n^4}{(2n-1)(2n+1)}.$$

42. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression Prove the identity

$$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\ldots+\frac{1}{a_na_1}=\frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right).$$

43. Prove that

1° $$\frac{n}{(n+1)!}+\frac{n+1}{(n+2)!}+\ldots+\frac{n+p}{(n+p+1)!}=\frac{1}{n!}-\frac{1}{(n+p+1)!};$$

2° $$\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots+\frac{1}{(n+p+1)!}<\frac{1}{n}\left[\frac{1}{n!}-\frac{1}{(n+p+1)!}\right]$$

(n and p any positive integers).

44. Simplify the following expression

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\ldots+\frac{2^n}{x^{2^n}+1}.$$

45. Let $S_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$.

Prove that

$$\frac{n+p+1}{n-p+1}\left\{\frac{n-p}{n(p+1)}+\frac{n-p-1}{(n-1)(p+2)}+\ldots+\frac{1}{n(p+1)}\right\}=S_n-S_p.$$

46. Let

$$S_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

$$S_n'=\frac{n+1}{2}-\left\{\frac{1}{n(n-1)}+\frac{2}{(n-1)(n-2)}+\ldots+\frac{n-2}{2\cdot 3}\right\}.$$

Prove that $S_n'=S_n$.

47. Let S_k be the sum of the first k terms of an arithmetic progression. What must this progression be for the ratio $\frac{S_{kx}}{S_x}$ to be independent of x?

48. Given that $a_1, a_2, \ldots, a_n$ form an arithmetic progression. Find the following sum:

$$S=\sum_{i=1}^{i=n}\frac{a_ia_{i+1}a_{i+2}}{a_i+a_{i+2}}.$$

49. Find the sum

$$\frac{1}{\cos\alpha\cos(\alpha+\beta)}+\frac{1}{\cos(\alpha+\beta)\cos(\alpha+2\beta)}+\ldots+\frac{1}{\cos[\alpha+(n-1)\beta]\cos(\alpha+n\beta)}$$

50. Show that

$$\tan\alpha+\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}+\ldots+\frac{1}{2^{n-1}}\tan\frac{\alpha}{2^{n-1}}=\frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}}-2\cot 2\alpha.$$

51. Prove the following formulas

1° $$\sin a+\sin(a+h)+\ldots+\sin[a+(n-1)h]=\frac{\sin\frac{nh}{2}\sin\left(a+\frac{n-1}{2}h\right)}{\sin\frac{h}{2}};$$

2° $$\cos a+\cos(a+h)+\ldots+\cos[a+(n-1)h]=\frac{\sin\frac{nh}{2}\cos\left(a+\frac{n-1}{2}h\right)}{\sin\frac{h}{2}}.$$

52. Find the following sums

$$S=\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n},$$

$$S'=\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\ldots+\cos\frac{(n-1)\pi}{n}.$$

53. Show that

$$\frac{\sin\alpha+\sin 3\alpha+\ldots+\sin(2n-1)\alpha}{\cos\alpha+\cos 3\alpha+\ldots+\cos(2n-1)\alpha}=\tan n\alpha.$$

54. Compute the sums

$$S_n=\cos^2 x+\cos^2 2x+\ldots+\cos^2 2nx,$$

$$S''_n=\sin^2 x+\sin^2 2x+\ldots+\sin^2 2nx.$$

55. Prove that

$$\sum_{i=1}^{i=p} \sin\frac{m\pi i}{p+1}\sin\frac{n\pi i}{p+1}=\begin{cases} -\frac{p+1}{2} & \text{if } m+n \text{ is divisible by } 2(p+1); \\ \frac{p+1}{2} & \text{if } m-n \text{ is divisible by } 2(p+1); \\ 0 & \text{if } m\neq n \text{ and if } m+n \text{ and } m-n \text{ are not divisible by } 2(p+1). \end{cases}$$

56. Find the sum

$$\arctan\frac{x}{1+1\cdot 2x^2}+\arctan\frac{x}{1+2\cdot 3x^2}+\ldots+\arctan\frac{x}{1+n(n+1)x^2}\quad (x>0).$$

57. Find the sum

$$\arctan\frac{r}{1+a_1a_2}+\arctan\frac{r}{1+a_2a_3}+\ldots+\arctan\frac{r}{1+a_na_{n+1}}$$

if a_1, a_2, ... form an arithmetic progression with a common difference r ($a_1>0$, $r>0$).

58. Compute the sum

$$\sum_{k=1}^{k=n}\arctan\frac{2k}{2+k^2+k^4}.$$

59. Solve the system

$$x_1\sin\frac{\pi}{n}+x_2\sin 2\frac{\pi}{n}+ x_3\sin 3\frac{\pi}{n}+\ldots+x_{n-1}\sin(n-1)\frac{\pi}{n}=a_1,$$

$$x_1\sin\frac{2\pi}{n}+x_2\sin 2\frac{2\pi}{n}+ x_3\sin 3\frac{2\pi}{n}+\ldots+x_{n-1}\sin(n-1)\frac{2\pi}{n}=a_2,$$

$$x_1 \sin\frac{3\pi}{n} + x_2 \sin 2\frac{3\pi}{n} +$$
$$+ x_3 \sin 3\frac{3\pi}{n} + \dots + x_{n-1} \sin (n-1)\frac{3\pi}{n} = a_3,$$

. . . .

$$x_1 \sin\frac{(n-1)\pi}{n} + x_2 \sin 2\frac{(n-1)\pi}{n} + x_3 \sin 3\frac{(n-1)\pi}{n} + \dots +$$
$$+ x_{n-1} \sin (n-1)\frac{(n-1)\pi}{n} = a_{n-1}.$$

8. INEQUALITIES

Let us recall the basic properties of inequalities.

1° If $a > b$ and $b > c$, then $a > c$.

2° If $a > b$, then $a + m > b + m$.

3° If $a > b$, then $am > bm$ for $m > 0$ and $am < bm$ for $m < 0$, i.e., when multiplying both members of the inequality by a negative number, the sign of the inequality is reversed.

4° If $a > b > 0$, then $a^x > b^x$ if $x > 0$.

This last inequality is readily proved for a rational x. Indeed, let us first assume that $x = m$ is a whole positive number. Then

$$a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \dots + b^{m-1}).$$

But either of the bracketed expressions on the right exceeds zero, therefore $a^m - b^m > 0$ and $a^m > b^m$. We now put $x = \frac{1}{m}$. Then $a^x - b^x = \sqrt[m]{a} - \sqrt[m]{b}$.

We have

$$(a-b) = (\sqrt[m]{a} - \sqrt[m]{b})(\sqrt[m]{a^{m-1}} + \dots + \sqrt[m]{b^{m-1}}).$$

Hence, actually, it follows that

$$\sqrt[m]{a} - \sqrt[m]{b} > 0, \text{ i.e. } \sqrt[m]{a} > \sqrt[m]{b}.$$

Let, finally, $x=\frac{p}{q}$. We have

$$a^x-b^x=a^{\frac{p}{q}}-b^{\frac{p}{q}}=\sqrt[q]{a^p}-\sqrt[q]{b^p}.$$

But $a^p > b^p$ (as has been proved), consequently, $\sqrt[q]{a^p} > \sqrt[q]{b^p}$. To prove this inequality for an irrational x we may consider x as a limit of a sequence of rational numbers and pass to the limit.

5° If $a>1$ and $x>y>0$, then $a^x>a^y$; but if $0<a<1$ and $x>y>0$, then $a^x<a^y$. The proof is basically reduced to that of $a^\alpha>1$ if $\alpha>0$ and $a>1$ and can be obtained from 4°.

6° $\log_a x > \log_a y$ if $x>y$ and $a>1$; and $\log_a x < \log_a y$ if $x>y$ and $0<a<1$.

Out of the problems considered in this section, utmost interest undoubtedly lies with Problem 30 both with respect to the methods of its solution and to the number of corollaries. Problem 50 should also be mentioned with its inequalities useful in many cases.

1. Show that

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{1}{2} \quad (n, \text{ a positive integer}).$$

2. Let n and p be positive integers and $n \geqslant 1$, $p \geqslant 1$. Prove that

$$\frac{1}{n+1}-\frac{1}{n+p+1}<\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+p)^2}<\frac{1}{n}-\frac{1}{n+p}.$$

3. Prove that the sum of any number of fractions taken from among the sequence $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots$ is always less than unity.

4. Prove that

$$\sqrt[n]{n!} \geqslant \sqrt{n}.$$

5. Show that if a is a defective value of $\sqrt{A}$ to within unity ($a<\sqrt{A}<a+1$), then

$$a+\frac{A-a^2}{2a+1}<\sqrt{A}<a+\frac{A-a^2}{2a+1}+\frac{1}{4(2a+1)}.$$

6. Prove that

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}<2\sqrt{n+1}-2.$$

7. Prove that

$$\frac{1}{2\sqrt{s}}<\frac{1}{4^s}C_{2s}^{s}<\frac{1}{\sqrt{2s+1}}.$$

8. Prove that

$$\cot\frac{\theta}{2}\geqslant 1+\cot\theta \quad (0<\theta<\pi).$$

9. Show that if $A+B+C=\pi$ $(A, B, C>0)$ and the angle C is obtuse, then

$$\tan A\tan B<1.$$

10. Let $\tan\theta=n\tan\varphi$ $(n>0)$.
Prove that

$$\tan^2(\theta-\varphi)\leqslant\frac{(n-1)^2}{4n}.$$

11. Show that if

$$\frac{1}{\cos\alpha\cos\beta}+\tan\alpha\tan\beta=\tan\gamma, \text{ then } \cos 2\gamma\leqslant 0.$$

12. Let us have n fractions

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}, b_i>0 \quad (i=1, 2, \ldots, n).$$

Prove that the fraction $\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$ is contained between the greatest and the least of these fractions.

13. Prove that $\sqrt[m+n+\ldots+p]{ab\ldots l}$ is contained between the greatest and the least one of the quantities

$$\sqrt[m]{a}, \sqrt[n]{b}, \ldots, \sqrt[p]{l}.$$

14. Suppose $0<\alpha<\beta<\gamma<\ldots<\lambda<\frac{\pi}{2}$.
Prove that

$$\tan\alpha<\frac{\sin\alpha+\sin\beta+\sin\gamma+\ldots+\sin\lambda}{\cos\alpha+\cos\beta+\cos\gamma+\ldots+\cos\lambda}<\tan\lambda.$$

15. Let $x^2 = y^2 + z^2$ $(x, y, z > 0)$.
Prove that

$$x^\lambda > y^\lambda + z^\lambda \text{ if } \lambda > 2,$$
$$x^\lambda < y^\lambda + z^\lambda \text{ if } \lambda < 2.$$

16. Prove that if

$$a^2 + b^2 = 1, \quad m^2 + n^2 = 1,$$

then $|am + bn| \leqslant 1$.

17. Let a, b, c and $a + b - c$, $a + c - b$, $b + c - a$ be positive.
Prove that

$$abc \geqslant (a + b - c)(a + c - b)(b + c - a).$$

18. Let

$$A + B + C = \pi.$$

Prove that

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 1.$$

19. Let

$$A + B + C = \pi \, (A, B, C > 0).$$

Prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leqslant \frac{1}{8}.$$

20. Given

$$A + B + C = \pi \, (A, B, C > 0).$$

Prove that

1° $\cos A + \cos B + \cos C \leqslant \frac{3}{2}$;

2° $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leqslant \frac{3\sqrt{3}}{8}$.

21. Prove that

$$\sqrt{(a+c)(b+d)} \geqslant \sqrt{ab} + \sqrt{cd} \quad (a, b, c \text{ and } d > 0).$$

22. Prove that

$$\frac{a^3+b^3}{2} \geqslant \left(\frac{a+b}{2}\right)^3 \quad (a > 0, \ b > 0).$$

23. Prove that

1° $\frac{a+b}{2} \geqslant \sqrt{ab} \quad (a\ b > 0);$

2° $\frac{1}{8}\frac{(a-b)^2}{a} \leqslant \frac{a+b}{2} - \sqrt{ab} \leqslant \frac{1}{8}\frac{(a-b)^2}{b}$ if $a \geqslant b$.

24. Prove that

$$\frac{a+b+c}{3} \geqslant \sqrt[3]{abc} \quad (a,\ b,\ c > 0).$$

25. Prove that

$$\sqrt{a_1a_2} + \sqrt{a_1a_3} + \ldots + \sqrt{a_{n-1}a_n} \leqslant \frac{n-1}{2}(a_1 + a_2 + \ldots + a_n)$$

$$(a_i > 0;\ i = 1,\ 2,\ \ldots,\ n).$$

26. Let $a_i > 0$ $(i = 1, 2, \ldots, n)$ and $a_1a_2 \ldots a_n = 1$. Prove that

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geqslant 2^n.$$

27. Prove that

1° $(a+b)(a+c)(b+c) \geqslant 8abc \quad (a,\ b,\ c > 0);$

2° $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geqslant \frac{3}{2}.$

28. Prove that

$$\sqrt[3]{(a+k)(b+l)(c+m)} \geqslant \sqrt[3]{abc} + \sqrt[3]{klm}$$

$$(a,\ b,\ c,\ k,\ l,\ m > 0).$$

29. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geqslant \frac{9}{a+b+c} \quad (a,\ b,\ c > 0).$$

30. Prove that

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1x_2 \ldots x_n} \quad (x_i > 0;\ i = 1,\ 2,\ \ldots,\ n),$$

the equality being obtained only in the case

$$x_1 = x_2 = \ldots = x_n.$$

31. Let $a_1, a_2, \ldots, a_n$ form an arithmetic progression $(a_i > 0)$.

Prove that $\sqrt{a_1a_n} \leqslant \sqrt[n]{a_1a_2\ldots a_n} \leqslant \frac{a_1+a_n}{2}$.
In particular

$$\sqrt{n} < \sqrt[n]{n!} < \frac{n+1}{2}.$$

32. Let a, b, and c be positive integers.
Prove that $a^{\frac{a}{a+b+c}} \cdot b^{\frac{b}{a+b+c}} \cdot c^{\frac{c}{a+b+c}} \geqslant \frac{1}{3}(a+b+c)$.

33. Prove that if a, b, c are positive, rational and such that the sum of every two numbers exceeds a third one, then

$$\left(1+\frac{b-c}{a}\right)^a\left(1+\frac{c-a}{b}\right)^b\left(1+\frac{a-b}{c}\right)^c \leqslant 1.$$

34. Let a, b, c, ..., l be n positive numbers and

$$s = a+b+c+\ldots+l.$$

Prove that $\frac{s}{s-a}+\frac{s}{s-b}+\ldots+\frac{s}{s-l} \geqslant \frac{n^2}{n-1}$.

35. Prove the inequality

$$(a_1b_1+a_2b_2+\ldots+a_nb_n)^2 \leqslant (a_1^2+a_2^2+\ldots+a_n^2)\times$$
$$\times(b_1^2+b_2^2+\ldots+b_n^2).$$

36. Prove the inequality

$$a_1+a_2+\ldots+a_n \leqslant \sqrt{n(a_1^2+a_2^2+\ldots+a_n^2)}.$$

37. Prove that

$$(x_1+x_2+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right) \geqslant n^2.$$

38. Let

$$x_1+x_2+\ldots+x_n = p,$$
$$x_1x_2+x_1x_3+\ldots+x_1x_n+x_2x_3+\ldots+x_{n-1}x_n = q.$$

Prove that

$$\frac{p}{n}+\frac{n-1}{n}\sqrt{p^2-\frac{2n}{n-1}q} \geqslant x_i \geqslant \frac{p}{n}-\frac{n-1}{n}\sqrt{p^2-\frac{2n}{n-1}q}.$$

39. Let a, b, c, ..., l be n real positive numbers and let p and q be also two real numbers.

Prove that if p and q are of the same sign, then

$$n(a^{p+q}+b^{p+q}+\ldots+l^{p+q})\geqslant(a^p+b^p+\ldots+l^p)\times$$
$$\times(a^q+b^q+\ldots+l^q).$$

And if p and q have different signs, then

$$n(a^{p+q}+b^{p+q}+\ldots+l^{p+q})\leqslant(a^p+b^p+\ldots+l^p)\times$$
$$\times(a^q+b^q+\ldots+l^q).$$

40. Prove that

1° $(1+\alpha)^\lambda > 1+\alpha\lambda$ (α is any positive number; $\lambda > 1$ is rational).

2° $(1+\alpha)^\lambda < \frac{1}{1-\alpha\lambda}$ ($\alpha > 0$ real, λ rational and positive, $\alpha\lambda < 1$).

41. Let $u_n = \left(1+\frac{1}{n}\right)^n$, n is a positive integer.

1° Prove that

$$u_{n+1} > u_n.$$

2° Prove that u_n is a bounded quantity, i.e. there exists a constant (independent of n) such that u_n is less than this constant for any n.

42. Prove that

$$\sqrt{2}>\sqrt[3]{3}>\sqrt[4]{4}>\sqrt[5]{5}>\sqrt[6]{6}>\ldots>\sqrt[n]{n}>$$
$$>\sqrt[n+1]{n+1}>\ldots .$$

43. Prove that

$$2>\sqrt{3}>\sqrt[3]{4}>\sqrt[4]{5}>\ldots>\sqrt[n-1]{n}>\sqrt[n]{n+1}>\ldots .$$

44. Let us have

$$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=y_1$$
$$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=y_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=y_n,$$

where $a_{ij} > 0$ and rational, $x_{ij} > 0$.

Furthermore, it is given that

$$a_{k1} + a_{k2} + \ldots + a_{kn} = 1,$$
$$a_{1k} + a_{2k} + \ldots + a_{nk} = 1 \quad (k = 1, 2, \ldots, n).$$

Prove that

$$y_1y_2 \ldots y_n \geqslant x_1x_2 \ldots x_n.$$

45. Let

$$a_i > 0, \ b_i > 0 \quad (i = 1, 2, \ldots, n).$$

Prove that

$$\sqrt[n]{(a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n)} \geqslant \sqrt[n]{a_1a_2 \ldots a_n} + \sqrt[n]{b_1b_2 \ldots b_n}.$$

46. Prove that

$$\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^k \leqslant \frac{x_1^k + x_2^k + \ldots + x_n^k}{n},$$

n and k are positive integers, $x_i > 0$.

47. Let the function $\varphi(t)$ defined in a certain interval possess the following property

$$\varphi\left(\frac{t_1 + t_2}{2}\right) < \frac{\varphi(t_1) + \varphi(t_2)}{n}$$

for any two t_1 and t_2 not equal to each other.

Then

$$\varphi\left(\frac{t_1 + t_2 + \ldots + t_n}{n}\right) < \frac{\varphi(t_1) + \varphi(t_2) + \ldots + \varphi(t_n)}{n},$$

where $t_1, t_2, \ldots, t_n$ are n arbitrary values from the given interval not equal to one another.

48. Find the greatest value of the sum

$$S = \sin a_1 + \sin a_2 + \ldots + \sin a_n$$

if

$$a_i > 0 \text{ and } a_1 + a_2 + \ldots + a_n = \pi.$$

49. Let x, p and q be positive, p and q being integers. Prove that

$$\frac{x^p - 1}{p} > \frac{x^q - 1}{q}$$

if $p > q$ $(x \neq 1)$.

50. Let $x > 0$ and not equal to 1, m rational. Prove that

$$mx^{m-1}(x-1) > x^m - 1 > m(x-1)$$

if m does not lie between 0 and 1.

But if $0 < m < 1$, then

$$mx^{m-1}(x-1) < x^m - 1 < m(x-1).$$

51. Prove that

$$(1+x)^m \geqslant 1 + mx$$

if m does not lie in the interval between 0 and 1;

$$(1+x)^m \leqslant 1 + mx$$

if $0 \leqslant m \leqslant 1$ (m rational, $x > -1$).

52. Prove that

$$\left(\frac{x_1^p + x_2^p + \ldots + x_n^p}{n}\right)^{\frac{1}{p}} \leqslant \left(\frac{x_1^q + x_2^q + \ldots + x_n^q}{n}\right)^{\frac{1}{q}},$$

$q \geqslant p$, both q and p being positive integers.

53. Find the value of x at which the expression

$$(x-x_1)^2 + (x-x_2)^2 + \ldots + (x-x_n)^2$$

takes on the least value.

54. Let $x_1 + x_2 + \ldots + x_n = C$ (C constant). At what $x_1, x_2, \ldots, x_n$ does the expression $x_1^2 + x_2^2 + \ldots + x_n^2$ attain the least value?

55. Let $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $x_1 + x_2 + \ldots + x_n = C$.

At what values of the variables $x_1, x_2, \ldots, x_n$ does the expression

$$x_1^\lambda + x_2^\lambda + \ldots + x_n^\lambda$$

(λ rational) attain the least value?

56. Given $x_i > 0$ $(i = 1, 2, \ldots, n)$ and the sum $x_1 + x_2 + \ldots + x_n = C = \text{const}$. Prove that the product $x_1x_2 \ldots x_n$ reaches the greatest value when $x_1 = x_2 = \ldots = x_n = \frac{C}{n}$.

57. Given $x_i > 0$ $(i = 1, 2, \ldots, n)$ and the product $x_1 x_2 x_3 \ldots x_n$ is constant, i.e., $x_1 x_2 \ldots x_n = C$.

Prove that the sum $x_1 + x_2 + \ldots + x_n$ attains the least value when

$$x_1 = x_2 = \ldots = x_n = \sqrt[n]{C}.$$

58. Let $x_i > 0$ $(i = 1, 2, \ldots, n)$ and the sum $x_1 + x_2 + \ldots + x_n = C = \text{const}$.

Show that

$$x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$$

takes on the greatest value when

$$\frac{x_1}{\mu_1} = \frac{x_2}{\mu_2} = \ldots = \frac{x_n}{\mu_n} = \frac{C}{\mu_1 + \mu_2 + \ldots + \mu_n},$$

$\mu_i > 0$ $(i = 1, 2, \ldots, n)$ and rational.

59. Let

$$a_i > 0, \; x_i > 0 \qquad (i = 1, 2, \ldots, n)$$

and

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = C.$$

Prove that the product $x_1 x_2 \ldots x_n$ attains the greatest value when

$$a_1 x_1 = a_2 x_2 = \ldots = a_n x_n = \frac{C}{n}.$$

60. Given

$$a_i > 0, \quad x_i > 0 \text{ and } a_1 x_1^{\lambda_1} + a_2 x_2^{\lambda_2} + \ldots + a_n x_n^{\lambda_n} = C$$

($\lambda_i > 0$ and rational).

Prove that

$$x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$$

takes on the greatest value when

$$\frac{\lambda_1 a_1 x_1^{\lambda_1}}{\mu_1} = \frac{\lambda_2 a_2 x_2^{\lambda_2}}{\mu_2} = \ldots = \frac{\lambda_n a_n x_n^{\lambda_n}}{\mu_n}.$$

61. Let $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} = C = \text{const}$.

Show that

$$a_1 x_1^{\mu_1} + a_2 x_2^{\mu_2} + \ldots + a_n x_n^{\mu_n}$$

attains the least value if

$$\frac{\frac{x_1^{\mu_1}}{\lambda_1}}{a_1\mu_1} = \frac{\frac{x_2^{\mu_2}}{\lambda_2}}{a_2\mu_2} = \ldots = \frac{\frac{x_n^{\mu_n}}{\lambda_n}}{a_n\mu_n}$$

($a_i, x_i > 0$; λ_i and $\mu_i > 0$ are rational).

62. Find at what values of $x, y, z, \ldots, t$ the sum

$$x^2 + y^2 + z^2 + \ldots + t^2$$

takes on the least value if

$$ax + by + \ldots + kt = A \quad (a, b, \ldots, k \text{ and } A \text{ constant}).$$

63. At what values of x, y does the expression

$$u = (a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 + \ldots + (a_nx + b_ny + c_n)^2$$

take on the least value?

64. Let $x_0, x_1, \ldots, x_n$ be integers and let us assume

$$x_0 < x_1 < x_2 < \ldots < x_n.$$

Prove that any polynomial of nth degree $x^n + a_1x^{n-1} + \ldots + a_n$ attains at points $x_0, x_1, \ldots, x_n$ the values at least one of which exceeds or equals $\frac{n!}{2^n}$.

65. Let $0 \leqslant x \leqslant \frac{\pi}{2}$. At what value of x does the product $\sin x \cos x$ reach the greatest value?

66. Let

$$x + y + z = \frac{\pi}{2}; \quad 0 \leqslant x \leqslant \frac{\pi}{2}, \quad 0 \leqslant y \leqslant \frac{\pi}{2}, \quad 0 \leqslant z \leqslant \frac{\pi}{2}.$$

At what values of x, y and z does the product $\tan x \tan y \times \tan z$ attain the greatest value?

67. Prove that

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1$$

(n a positive integer).

68. Let $a > 1$ and n be a positive integer. Prove that

$$a^n - 1 \geqslant n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right).$$

69. Prove that

$$\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n - 1} < n$$

(n a positive integer).

70. Prove that

$$\frac{1}{\frac{1}{a}+\frac{1}{b}} + \frac{1}{\frac{1}{c}+\frac{1}{d}} \leqslant \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}} \quad (a, b, c, d > 0).$$

9. MATHEMATICAL INDUCTION

This section contains problems which are mainly solved using the method of mathematical induction. A certain amount of problems is dedicated to combinatorics.

1. Given

$$v_{n+1} = 3v_n - 2v_{n-1}$$

and

$$v_0 = 2, \quad v_1 = 3.$$

Prove that

$$v_n = 2^n + 1.$$

2. Let

$$u_{n+1} = 3u_n - 2u_{n-1}$$

and

$$u_0 = 0, \quad u_1 = 1.$$

Prove that

$$u_n = 2^n - 1.$$

3. Let a and $A > 0$ be arbitrary given numbers and let

$$a_1 = \frac{1}{2}\left(a + \frac{A}{a}\right), \quad a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right), \quad \ldots, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{A}{a_{n-1}}\right).$$

Prove that

$$\frac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$$

for any whole n.

4. The series of numbers

$$a_0, a_1, a_2, \ldots$$

is formed according to the following law. The first two numbers a_0 and a_1 are given, each subsequent number being equal to the half-sum of two previous ones. Express a_n in terms of a_0, a_1 and n.

5. The terms of the series

$$a_1, a_2, a_3, \ldots$$

are determined as follows

$$a_1 = 2 \text{ and } a_n = 3a_{n-1} + 1.$$

Find the sum

$$a_1 + a_2 + \ldots + a_n.$$

6. The terms of the series

$$a_1, a_2, \ldots$$

are connected by the relation

$$a_n = ka_{n-1} + l \, (n = 2, 3, \ldots).$$

Express a_n in terms of a_1, k, l and n.

7. The sequence $a_1, a_2, \ldots$ satisfies the relation $a_{n+1} - 2a_n + a_{n-1} = 1$.

Express a_n in terms of a_1, a_2 and n.

8. The terms of the series

$$a_1, a_2, a_3, \ldots$$

are related in the following way $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 1$.

Express a_n in terms of a_1, a_2, a_3 and n.

9. Let the pairs of numbers

$$(a, b)\ (a_1, b_1)\ (a_2, b_2) \ldots$$

be obtained according to the following law

$$a_1 = \frac{a+b}{2}, \quad b_1 = \frac{a_1+b}{2}, \quad a_2 = \frac{a_1+b_1}{2}, \quad b_2 = \frac{a_2+b_1}{2}, \ldots .$$

Prove that

$$a_n = a + \frac{2}{3}(b-a)\left(1 - \frac{1}{4^n}\right),$$

$$b_n = a + \frac{2}{3}(b-a)\left(1 + \frac{1}{2\cdot 4^n}\right).$$

10. The terms of the series

$$x_0, y_0, x_1, y_1, x_2, y_2, \ldots$$

are determined by the relations

$$x_n = x_{n-1} + 2y_{n-1}\sin^2\alpha, \quad y_n = y_{n-1} + 2x_{n-1}\cos^2\alpha.$$

Besides, it is known that $x_0 = 0$, $y_0 = \cos\alpha$.

Express x_n and y_n in terms of α.

11. The numbers

$$x_0, x_1, x_2, \ldots, \quad y_0, y_1, y_2, \ldots$$

are related as follows

$$\begin{aligned} x_n &= \alpha x_{n-1} + \beta y_{n-1}, \\ y_n &= \gamma x_{n-1} + \delta y_{n-1} \end{aligned} \qquad (\alpha\delta - \beta\gamma \neq 0).$$

Express x_n and y_n in terms of x_0, y_0 and n.

12. The terms of the series

$$x_0, x_1, x_2, \ldots$$

are determined by the relation

$$x_n = \alpha x_{n-1} + \beta x_{n-2}.$$

Express x_n in terms of x_0, x_1 and n.

13. The terms of the series $x_0, x_1, \ldots$ are connected by the relation

$$x_n = \frac{px_{n-1} + qx_{n-2}}{p+q}.$$

Express x_n in terms of x_0, x_1 and n.

14. The terms $x_0, x_1, x_2, \ldots$ are determined by the equality

$$x_n = \frac{\alpha x_{n-1} + \beta}{\gamma x_{n-1} + \delta}.$$

Express x_n in terms of x_0 and n.
Consider the particular cases

$$x_n = \frac{x_{n-1}}{2x_{n-1}+1}, \quad x_n = \frac{x_{n-1}+1}{x_{n-1}+3}.$$

15. The numbers:

$$a_0, a_1, a_2, \ldots,$$
$$b_0, b_1, b_2, \ldots$$

are determined by the following law

$$a_{n+1} = \frac{a_n+b_n}{2}, \quad b_{n+1} = \frac{2a_nb_n}{a_n+b_n},$$

a_0 and b_0 are given, and $a_0 > b_0 > 0$. Express a_n and b_n in terms of a_0, b_0 and n.

16. Prove the identity

$$\frac{n}{2n+1} + \frac{1}{2^3-2} + \ldots + \frac{1}{(2n)^3-2n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

17. Simplify the expression

$$(1-x)(1-x^2)\ldots(1-x^n) + x(1-x^2)(1-x^3)\ldots \times (1-x^n) + x^2(1-x^3)\ldots(1-x^n) + \ldots + x^k(1-x^{k+1})\ldots(1-x^n) + \ldots + x^{n-1}(1-x^n) + x^n$$

18. Prove the identity

$$\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \ldots + \frac{x^{2^{n-1}}}{1-x^{2^n}} = \frac{1}{1-x} \cdot \frac{x-x^{2^n}}{1-x^{2^n}}.$$

19. Check the identity

$$(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^{n-1}}) = 1 + x + x^2 + x^3 + \ldots + x^{2^n-1}$$

20. Prove the validity of the identity

$$1 + \frac{1}{a} + \frac{a+1}{ab} + \frac{(a+1)(b+1)}{abc} + \ldots + \frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots skl} = \frac{(a+1)(b+1)\ldots(k+1)(l+1)}{abc\ldots kl}.$$

21. Prove the identity

$$\frac{b+c+d+\ldots+k+l}{a(a+b+c+\ldots+k+l)}=\frac{b}{a(a+b)}+\frac{c}{(a+b)(a+b+c)}+\ldots+$$
$$+\frac{d}{(a+b+c)(a+b+c+d)}+\ldots+$$
$$+\frac{l}{(a+b+\ldots+k)(a+b+\ldots+k+l)}.$$

22. Let

$$\frac{q}{1-q}(1-z)+\frac{q^2}{1-q^2}(1-z)(1-qz)+\ldots+$$
$$+\frac{q^n}{1-q^n}(1-z)(1-qz)\ldots(1-q^{n-1}z)=F_n(z).$$

Prove the identity

$$1+F_n(z)-F_n(qz)=(1-qz)(1-q^2z)\ldots(1-q^nz).$$

23. Prove that

$$\sum_{k=1}^{k=n}\frac{(1-a^n)(1-a^{n-1})\ldots(1-a^{n-k+1})}{1-a^k}=n.$$

24. Compute the sum

$$S_n=\frac{a}{b}+\frac{a(a-1)}{b(b-1)}+\frac{a(a-1)(a-2)}{b(b-1)(b-2)}+\frac{a(a-1)\ldots(a-n+1)}{b(b-1)\ldots(b-n+1)}$$

(b is not equal to $0, 1, 2, \ldots, n-1$).

25. Let

$$S_n=a_1+(a_1+1)a_2+(a_1+1)(a_2+1)a_3+\ldots+$$
$$+(a_1+1)(a_2+1)\ldots(a_{n-1}+1)a_n.$$

Prove that

$$S_n=(a_1+1)(a_2+1)\ldots(a_n+1)-1.$$

26. Prove the following identities:

1° $$\sum_{x=1}^{x=n}x(x+1)\ldots(x+q)=\frac{1}{q+2}n(n+1)\ldots(n+q+1);$$

2° $$\sum_{x=1}^{x=n}\frac{1}{x(x+1)\ldots(x+q)}=\frac{1}{q}\left\{\frac{1}{q!}-\frac{1}{(n+1)(n+2)\ldots(n+q)}\right\}.$$

27. Prove the identity

$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\dots+$$
$$+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)=$$
$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2n-1}-\frac{1}{2n}\right).$$

28. Let us have a sequence of numbers (Fibonacci's series)

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \dots .$$

This sequence is determined by the following conditions

$$u_{n+1}=u_n+u_{n-1}$$

and $u_0=0$, $u_1=1$.

Show that there exist the following relations

1° $u_{n+2}=u_0+u_1+u_2+\dots+u_n+1$;

2° $u_{2n+2}=u_1+u_3+u_5+\dots+u_{2n+1}$;

3° $u_{2n+1}=1+u_2+u_4+\dots+u_{2n}$;

4° $-u_{2n-1}+1=u_1-u_2+u_3+\dots+u_{2n-1}-u_{2n}$;

5° $u_{2n-2}+1=u_1-u_2+u_3-u_4+\dots+u_{2n-1}$;

6° $u_nu_{n+1}=u_1^2+u_2^2+\dots+u_n^2$;

7° $u_{2n}^2=u_1u_2+u_2u_3+\dots+u_{2n-1}u_{2n}$;

8° $u_{n+1}u_{n+2}-u_nu_{n+3}=(-1)^n$;

9° $u_n^2-u_{n+1}u_{n-1}=(-1)^{n+1}$;

10° $u_n^4-u_{n-2}u_{n-1}u_{n+1}u_{n+2}=1$.

29. Compute the sum

$$\frac{1}{1\cdot 2}+\frac{2}{1\cdot 3}+\dots+\frac{u_{n+2}}{u_{n+1}u_{n+3}}.$$

30. Prove the relations

1° $u_{n+p-1}=u_{n-1}u_{p-1}+u_nu_p$;

2° $u_{2n-1}=u_n^2+u_{n-1}^2$;

3° $u_{2n-1}=u_nu_{n+1}-u_{n-2}u_{n-1}$.

31. Prove that $u_n^3 + u_{n+1}^3 - u_{n-1}^3 = u_{3n}$.

32. Prove that $u_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} C_{n-k-1}^{k}$.

33. Find the number of whole positive solutions of the equation $x_1 + x_2 + \ldots + x_n = m$ (m a positive integer).

34. Prove that the total number of whole nonnegative solutions of the equations

$$x + 2y = n, \quad 2x + 3y = n - 1, \ldots, nx + (n + 1)\, y = 1,$$
$$(n + 1)\, x + (n + 2)\, y = 0$$

is equal to $n + 1$.

35. Show that the total number of whole nonnegative solutions of the equations

$$x + 4y = 3n - 1, \quad 4x + 9y = 5n - 4, \quad 9x + 16y = 7n - 9, \ldots, n^2x + (n + 1)^2\, y = n\,(n + 1)$$

is equal to n.

36. There are n white and n black balls marked $1, 2, 3, \ldots, n$. In how many ways can the balls be arranged in a row so that all neighbouring balls were of different colour?

37. In how many ways is it possible to distribute kn distinct objects into k groups, each consisting of n elements?

38. How many permutations can be made up of n elements in which the two elements a and b never stand side by side?

39. Find the number of permutations of n elements in which none of the elements occupies the original position.

40. In how many ways can n distinct letters be arranged in r squares (first, second, ..., rth square) so that each square contains at least one letter (the order of the letters inside the squares is disregarded)?

10. LIMITS

We take as known the concept of a variable and its limit, as well as the basic theorems on limits which are usually treated in elementary textbooks of algebra (the limit of a sum, product and quotient). Let us here remind the reader of one of the indications for a limit to exist: if a variable increases but remains smaller than a certain constant, then such a variable has a limit (likewise, a variable which, when decreasing, remains greater than a certain constant also has a limit). When dealing with an infinitely decreasing geometric progression and, in general, with simple infinite series, one should bear in mind that the symbolic notation

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

denotes none other than $\lim\limits_{n\to\infty} (u_1 + u_2 + \ldots + u_n)$ if such a limit exists. If there is no limit, then the series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is said to be divergent, and it is useless to speak of its numerical value.

1. Let $x_n = a^n$ and $|a| < 1$. Prove that $\lim\limits_{n\to\infty} x_n = 0$.

2. Prove that

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

for any real a.

3. Find

$$\lim_{n\to\infty} \frac{a_0 n^k + a_1 n^{k-1} + \ldots + a_k}{b_0 n^h + b_1 n^{h-1} + \ldots + b_h}$$

$$(a_0 \neq 0, \quad b_0 \neq 0).$$

4. Let

$$P_n = \frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \ldots \frac{n^3-1}{n^3+1}.$$

Prove that $\lim\limits_{n\to\infty} P_n = \frac{2}{3}$.

5. Prove that

$$\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} = \frac{1}{k+1} \quad (k \text{ a positive integer}).$$

6. Prove that

$$\lim_{n\to\infty} \left\{ \frac{1^k + 2^k + \ldots + n^k}{n^k} - \frac{n}{k+1} \right\} = \frac{1}{2}$$

(k a positive integer).

7. Let us have a sequence of numbers x_n determined by the equality

$$x_n = \frac{x_{n-1} + x_{n-2}}{3}$$

and the values x_0 and x_1.

Prove that

$$\lim_{n \to \infty} x_n = \frac{x_0 + 2x_1}{3}.$$

8. Let $N > 0$. Let us take an arbitrary positive number x_0 and form the following sequence

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right),$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right),$$

$$\cdots\cdots\cdots\cdots$$

$$x_p = \frac{1}{2}\left(x_{p-1} + \frac{N}{x_{p-1}}\right),$$

$$\cdots\cdots\cdots\cdots$$

Prove that $\lim\limits_{n \to \infty} x_n = \sqrt{N}$.

9. Generalize the result of the preceding problem for the extracting a root of any index from a positive number.

Prove that if

$$x_1 = \frac{m-1}{m} x_0 + \frac{N}{m x_0^{m-1}},$$

$$x_2 = \frac{m-1}{m} x_1 + \frac{N}{m x_1^{m-1}},$$

$$\cdots\cdots\cdots\cdots$$

$$x_p = \frac{m-1}{m} x_{p-1} + \frac{N}{m x_{p-1}^{m-1}},$$

$$\cdots\cdots\cdots\cdots$$

then

$$\lim_{n \to \infty} x_n = \sqrt[m]{N}.$$

10. Prove that

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0.$$

11. Let

$$S_n = \sum_{k=1}^{k=n} \left(\sqrt{1+\frac{k}{n^2}} - 1\right).$$

Find

$$\lim_{n\to\infty} S_n.$$

12. Let the variable x_n be determined by the following law of formation

$$x_0 = \sqrt{a},$$

$$x_1 = \sqrt{a+\sqrt{a}},$$

$$x_2 = \sqrt{a+\sqrt{a+\sqrt{a}}},$$

$$x_3 = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Find

$$\lim_{n\to\infty} x_n.$$

13. Prove that the variable

$$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

has a limit as $n \to \infty$.

14. Let us be given two sequences

$$x_0, x_1, x_2, \ldots,$$

$$y_0, y_1, y_2, \ldots \quad (x_0 > y_0 > 0),$$

where each subsequent term is formed from the preceding ones in the following manner

$$x_n = \frac{x_{n-1}+y_{n-1}}{2}, \quad y_n = \sqrt{x_{n-1}y_{n-1}}.$$

Prove that x_n and y_n have limits which are equal to each other.

15. Let

$$S_1 = 1 + q + q^2 + \ldots \quad |q| < 1,$$
$$S = 1 + Q + Q^2 + \ldots \quad |Q| < 1.$$

Find

$$1 + qQ + q^2Q^2 + \ldots .$$

16. Let s be the sum of terms of an infinite geometric progression, σ^2 the sum of squares of the terms. Show that the sum of n terms of this progression is equal to

$$s\left\{1 - \left[\frac{s^2-\sigma^2}{s^2+\sigma^2}\right]^n\right\}.$$

17. Prove that

1° $\lim\limits_{n\to\infty} n^k x^n = 0$ if $|x| < 1$ and k is a positive integer;

2° $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

18. Find the sums of the following series

1° $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots;$

2° $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} + \ldots.$

19. Prove that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$$

is a divergent one.

20. Prove that the series

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \frac{1}{4^\alpha} + \ldots + \frac{1}{n^\alpha} + \ldots$$

is a convergent one if $\alpha > 1$.

21. Find the sums of the following series

1° $1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots;$

2° $1 + 4x + 9x^2 + \ldots + n^2x^{n-1} + \ldots;$

3° $1 + 2^3x + 3^3x^2 + \ldots + n^3x^{n-1} + \ldots \quad (|x| < 1).$

22. 1° Prove that the variable $u_n = \left(1 + \frac{1}{n}\right)^n$ $(n = 1, 2, 3, \ldots)$ has a limit.

2° Denoting the limit u_n by e so that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, prove that

$$e = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \ldots + \frac{1}{1\cdot 2\cdot 3\ldots k} + \frac{\theta}{1\cdot 2\cdot 3\ldots k\cdot k}$$

$$(0 < \theta < 1).$$

23. Let $0 < x < \frac{\pi}{2}$.

Knowing that $\lim_{x\to 0}\frac{\sin x}{x} = 1$, prove that

$$x - \sin x \leqslant \frac{1}{6}x^3.$$

24. 1° Prove that the series

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \ldots + \frac{a_n}{10^n} + \ldots \quad (0 \leqslant a_i \leqslant 9)$$

is a convergent one.

2° Prove that for any real number ω $(0 < \omega < 1)$ it is always possible to find, and in the unique way, a_i $(0 \leqslant a_i \leqslant 9$; a_i being integers), such that

$$\omega = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \ldots + \frac{a_n}{10^n} + \ldots$$

(i.e. to expand the real number in decimal fractions).

3° Show that if a decimal fraction

$$\omega = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \ldots + \frac{a_n}{10^n} + \ldots$$

is finite or periodic (i.e., for instance, $a_{n+1} = a_1$, $a_{n+2} = a_2, \ldots, a_{2n} = a_n, \ldots$, so that the period contains n digits: $a_1, a_2, \ldots, a_n$), then ω is a rational number.

25. Prove that the numbers determined by the following series are irrational ones

1° $\omega = \frac{1}{l} + \frac{1}{l^4} + \frac{1}{l^9} + \frac{1}{l^{16}} + \ldots + \frac{1}{l^{n^2}} + \ldots,$

where l is any positive integer.

$$2^\circ\quad \omega=\frac{1}{l}+\frac{1}{l^{1\cdot 2}}+\frac{1}{l^{1\cdot 2\cdot 3}}+\frac{1}{l^{1\cdot 2\cdot 3\cdot 4}}+$$

$+\ldots+\frac{1}{l^{1\cdot 2\cdot 3\ldots n}}+\ldots$, where l is any positive integer.

26. Prove that e is an irrational number (see Problem 22).

27. Let

$$\omega=\frac{1}{l_1}+\frac{1}{l_1l_2}+\frac{1}{l_1l_2l_3}+\ldots+\frac{1}{l_1l_2\ldots l_n}+\ldots,$$

where $1<l_1\leqslant l_2\leqslant l_3\ldots$ and l_i are integers. Prove that ω s rational only when l_k (beginning with a certain k) are all equal to one another.

28. Prove that the variable

$$u_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$$

has a limit.

29. Prove the following formula:

$$\frac{\pi}{2}=\frac{1}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\ldots}}}}.$$

SOLUTIONS

SOLUTIONS TO SECTION 1

1. Proved immediately by a check.

2. If we remove the brackets from the right member and apply the formula for a square of a polynomial, then it is easily seen that all the doubled products are cancelled out, and we get the required identity.

3. If the identity of the preceding problem is used, then from the condition of our problem it follows that

$$(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + t^2) = 0,$$

whence either $a^2 + b^2 + c^2 + d^2 = 0$, or $x^2 + y^2 + z^2 + t^2 = 0$.

But the sum of the squares of real numbers equals zero only when each of the numbers is equal to zero. Therefore, from the equality $a^2 + b^2 + c^2 + d^2 = 0$, we get $a = b = c = d = 0$, and from the equality $x^2 + y^2 + z^2 + t^2 = 0$ we have $x = y = z = t = 0$.

Hence follows the required result.

4. This identity can be checked directly, and also can be obtained from identity (2) if we put in it $d = t = 0$ and replace y by $-y$ and z by $-z$.

5. If we expand the right member of the equality, then all doubled products are cancelled out and the validity of the identity becomes obvious.

6. Put in identity (5) $a_1 = a_2 = a_3 = \ldots = a_n = 1$, $b_1 = a$, $b_2 = b, \ldots, b_{n-1} = k$, $b_n = l$.

We then get

$$n(a^2 + b^2 + c^2 + \ldots + k^2 + l^2) = \\ = (a + b + \ldots + l)^2 + (b - a)^2 + \\ + (c - a)^2 + \ldots + (k - l)^2.$$

But since by hypothesis

$$n\,(a^2 + b^2 + \ldots + k^2 + l^2) = (a + b + \ldots + k + l)^2,$$

we have

$$(b - a)^2 + (c - a)^2 + \ldots + (k - l)^2 = 0.$$

Hence $a = b = c = \ldots = k = l$.

7. Make use of identity (5). By hypothesis

$$a_1^2 + a_2^2 + \ldots + a_n^2 = 1, \quad b_1^2 + b_2^2 + \ldots + b_n^2 = 1.$$

Therefore we have

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 = \\ = 1 - (a_1b_2 - a_2b_1)^2 - (a_1b_3 - a_3b_1)^2 - \ldots - \\ - (a_{n-1}b_n - a_nb_{n-1})^2.$$

Whence

$$0 \leqslant (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant 1.$$

Thus,

$$-1 \leqslant a_1b_1 + a_2b_2 + \ldots + a_nb_n \leqslant +1.$$

8. We have

$$(y + z - 2x)^2 - (y - z)^2 + (z + x - 2y)^2 - (z - x)^2 + \\ + (x + y - 2z)^2 - (x - y)^2 = 0.$$

But

$$(y + z - 2x)^2 - (y - z)^2 = 4\,(y - x)\,(z - x)$$

(using the formula for a difference of squares).

Likewise we find

$$(z + x - 2y)^2 - (z - x)^2 = 4\,(z - y)\,(x - y),$$
$$(x + y - 2z)^2 - (x - y)^2 = 4\,(x - z)\,(y - z).$$

Consequently,

$$4\,(y - x)\,(z - x) + 4\,(z - y)\,(x - y) + \\ + 4\,(x - z)\,(y - z) = 0$$

Removing the brackets, we get

$$2x^2 + 2y^2 + 2z^2 - 2xz - 2yz - 2xy = 0$$

or

$$(x-y)^2+(x-z)^2+(y-z)^2=0,$$

whence

$$x=y=z=0.$$

9. The first identity is obvious. Let us rewrite the second one in the following way

$$(6a^2-4ab+4b^2)^3-(4a^2-4ab+6b^2)^3=$$
$$=(3a^2+5ab-5b^2)^3+(5a^2-5ab-3b^2)^3.$$

Applying the formula for a difference of cubes to the left member and the formula for a sum of cubes to the right member, we find that it suffices to prove the following identity

$$(3a^2-2ab+2b^2)^2+(3a^2-2ab+2b^2)(2a^2-2ab+3b^2)+$$
$$+(2a^2-2ab+3b^2)^2=(5a^2-5ab-3b^2)^2-$$
$$-(5a^2-5ab-3b^2)(3a^2+5ab-5b^2)+$$
$$+(3a^2+5ab-5b^2)^2.$$

This identity is proved by directly removing the brackets.

10. To see whether the identity under consideration is valid, we may rewrite it as

$$(p^2-q^2)^4=(p^2+pq+q^2)^4-(2pq+q^2)^4+$$
$$+(p^2+pq+q^2)^4-(2pq+p^2)^4.$$

It remains to simplify the right member and to show that it is equal to the left one.

Using the formula $A^4-B^4=(A+B)(A-B)(A^2+B^2)$, we get the following expression for the right member

$$(p^2+3pq+2q^2)(p^2-pq)[(p^2+pq+q^2)^2+$$
$$+(2pq+q^2)^2]+(2p^2+3pq+q^2)(q^2-pq)\times$$
$$\times[(p^2+pq+q^2)^2+(2pq+p^2)^2]=(p+2q)\times$$
$$\times p(p^2-q^2)[(p^2+pq+q^2)^2+(2pq+q^2)^2]+$$
$$+(2p+q)q(q^2-p^2)[(p^2+pq+q^2)^2+$$
$$+(2pq+p^2)^2]=(p^2-q^2)\{(p^2+pq+q^2)^2\times$$
$$\times[p^2+2pq^2-2pq-q^2]+(p^2+2pq)(q^2+2pq)\times$$
$$\times[2pq+q^2-2pq-p^2]\}=(p^2-q^2)^2\{(p^2+pq+q^2)^2-$$
$$-(p^2+2pq)(q^2+2pq)\}=(p^2-q^2)^4.$$

11. Check by direct substitution.

12. Check by substitution.

13. 1° The cases $n = 0, 1, 2$ are readily checked directly. At $n = 4$ let us rewrite the identity in the following way

$$(ix - ky)^4 - (ix - kz)^4 + (iy - kz)^4 - \\ - (iy - kx)^4 + (iz - kx)^4 - \\ - (iz - ky)^4 = 0.$$

Transform the first two terms

$$(ix - ky)^4 - (ix - kz)^4 = [(ix - ky)^2 + \\ + (ix - kz)^2] (2ix - ky - kz) k (z - y). \quad (1)$$

By virtue of the equality $x + y + z = 0$, we get

$$2ix - ky - kz = (2i + k) x.$$

The expression in square brackets can be rewritten as follows

$$(2i^2 + 2ik) x^2 + k^2 (y^2 + z^2).$$

Thus, we have

$$(ix - ky)^4 - (ix - kz)^4 = \\ = k (2i + k) (y^2 - z^2) [(2i^2 + 2ik) x^2 + k^2 (y^2 + z^2)]. \quad (1')$$

It remains to transform the following expressions

$$(iy - kz)^4 - (iy - kx)^4, \quad (2)$$
$$(iz - kx)^4 - (iz - ky)^4. \quad (3)$$

But it is easily seen that expression (2) is obtained from the first one, already considered, by means of a circular permutation of the letters x, y and z, i.e. when x is replaced by y, y by z, and z by x. Expression (3) is obtained from (2) also through such a permutation. Therefore, there is no need to repeat computations for simplifying expressions (2) and (3); it is sufficient only to apply appropriate permutations to the result obtained. We then have

$$(iy - kz)^4 - (iy - kx)^4 = \\ = k (2i + k) (z^2 - x^2) [(2i^2 + 2ik) y^2 + \\ + k^2 (z^2 + x^2)], \quad (2')$$

$$(iz - kx)^4 - (iz - ky)^4 = \\ = k(2i + k)(x^2 - y^2)[(2i^2 + 2ik)z^2 + \\ + k^2(x^2 + y^2)]. \qquad (3')$$

And adding expressions (1′), (2′) and (3′), we get

$$k(2i + k)\{(2i^2 + 2ik)[(y^2 - z^2)x^2 + (z^2 - x^2)y^2 + \\ + (x^2 - y^2)z^2] + \\ + k^2(y^4 - z^4 + z^4 - x^4 + \\ + x^4 - y^4)\} = 0.$$

2° At $n = 0$ the relation is obvious. Let us denote, for brevity, the sum in the left member of the equality by

$$\sum (x + k)^n,$$

and the sum in the right member by

$$\sum (x + l)^n.$$

At $n = 1$ we have to prove that

$$8x + \sum k = 8x + \sum l,$$

i.e. we have to prove that

$$\sum k = \sum l.$$

Finally, we have to check that

$$\sum k = \sum l.$$

But

$$\sum k = 3 + 5 + 6 + 9 + 10 + 12 + 15 = 60,$$
$$\sum l = 1 + 2 + 4 + 7 + 8 + 11 + 13 + 14 = 60.$$

At $n = 2$ we have to prove that

$$\sum (x + k)^2 = \sum (x + l)^2,$$

i.e. that

$$8x^2 + 2x\sum k + \sum k^2 = 8x^2 + 2x\sum l + \sum l^2.$$

And so, it remains to prove that

$$\sum k^2 = \sum l^2,$$

which is easily checked directly.

Likewise, to prove the last case ($n = 3$) we have only to show that

$$\sum k^3 = \sum l^3.$$

14. The first identity is proved in the following way

$$(a + b + c + d)^2 + (a + b - c - d)^2 +$$
$$+ (a + c - b - d)^2 + (a + d - b - c)^2 =$$
$$= [(a + b) + (c + d)]^2 + [(a + b) - (c + d)]^2 +$$
$$+ [(a - b) + (c - d)]^2 + [(a - b) - (c - d)]^2 =$$
$$= 2(a + b)^2 + 2(c + d)^2 + 2(a - b)^2 + 2(c - d)^2 =$$
$$= 2[(a + b)^2 + (a - b)^2] + 2[(c + d)^2 +$$
$$+ (c - d)^2] = 4(a^2 + b^2 + c^2 + d^2).$$

The second and third identities are also proved by a direct check with some preliminary transformations.

15. Rewrite our equality as follows

$$[(a + b + c)^4 - (a^4 + b^4 + c^4)] + [(b + c - a)^4 -$$
$$- (a^4 + b^4 + c^4)] + [(c + a - b)^4 -$$
$$- (a^4 + b^4 + c^4)] + [(a + b - c)^4 -$$
$$- (a^4 + b^4 + c^4)] = 24(a^2b^2 + a^2c^2 + b^2c^2).$$

Consider the first term.

We have

$$(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)^2 - a^4 - b^4 - c^4 =$$
$$= 6a^2b^2 + 6a^2c^2 + 6b^2c^2 + 4ac(a^2 + c^2) +$$
$$+ 4ab(a^2 + b^2) + 4bc(b^2 + c^2) + 12a^2bc +$$
$$+ 12b^2ac + 12c^2ab.$$

The remaining terms are obtained from the first one by means of successive substitutions: $-a$ for a, $-b$ for b, $-c$ for c. Adding the terms, we make sure that our identity is valid.

16. We have

$$s(s-2b)(s-2c)+s(s-2c)(s-2a)+ \\ +s(s-2a)(s-2b)=(s-2a)(s-2b)(s-2c)+ \\ +2a(s-2b)(s-2c)+s(s-2a)(2s-2c-2b)= \\ =(s-2a)(s-2b)(s-2c)+2a(s-2b)(s-2c)+ \\ +s(s-2a)\,2a.$$

Transform the sum

$$2a(s-2b)(s-2c)+s(s-2a)\,2a= \\ =2a[(s-2b)(s-2c)+s(s-2a)]= \\ =2a[(s-2b)(s-2c)+(s-2a)(s-2b)+ \\ +2b(s-2a)]=2a[(s-2b)(2s-2c-2a)+ \\ +2b(s-2a)]=2a[(s-2b)\,2b+2b(s-2a)]= \\ =2a\cdot 2b\,[s-2b-2a]=4ab\cdot 2c=8abc.$$

17. Expanding the expression in the left member in powers of s, we get

$$(a+b+c)\,s^2-2s(a^2+b^2+c^2)+a^3+b^3+c^3+ \\ +2s^3-2s^2(a+b+c)+ \\ +2s(ab+ac+bc)-2abc.$$

Since $a+b+c=2s$, we have

$$2s^3-2s(a^2+b^2+c^2)+a^3+b^3+c^3+2s^3-4s^3+ \\ +2s(ab+ac+bc)-2abc=-2s(a^2+b^2+c^2)+ \\ +a^3+b^3+c^3+2s(ab+ac+bc)-2abc= \\ =a^3+b^3+c^3+(a+b+c)(ab+ac+bc- \\ -a^2-b^2-c^2)-2abc.$$

Directly transforming the last expression, we make sure that it is equal to abc (see also Problem 20).

18. We have

$$(2\sigma^2-2a^2)(2\sigma^2-2b^2)=(a^2+c^2-b^2)(b^2+c^2-a^2)= \\ =c^4-(a^2-b^2)^2.$$

Using a circular permutation, we obtain

$$(2\sigma^2 - 2b^2)(2\sigma^2 - 2c^2) = a^4 - (b^2 - c^2)^2,$$
$$(2\sigma^2 - 2c^2)(2\sigma^2 - 2a^2) = b^4 - (c^2 - a^2)^2.$$

Hence

$$4[(\sigma^2 - a^2)(\sigma^2 - b^2) + (\sigma^2 - b^2)(\sigma^2 - c^2) + (\sigma^2 - c^2)(\sigma^2 - a^2)] = a^4 + b^4 + c^4 - (a^2 - b^2)^2 - (b^2 - c^2)^2 - (c^2 - a^2)^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 = -[a^4 - 2(b^2 + c^2)a^2 + (b^2 - c^2)^2] = -[a^4 - 2(b^2 - c^2)a^2 + (b^2 - c^2)^2 - 4a^2c^2] = 4a^2c^2 - (a^2 - b^2 + c^2)^2 = (2ac + a^2 - b^2 + c^2)(2ac - a^2 + b^2 - c^2) = (a + b + c)(a + c - b)(b - a + c)(b + a - c).$$

But

$$a + b + c = 2s, \quad a + b - c = 2(s - c),$$
$$a + c - b = 2(s - b), \quad b + c - a = 2(s - a)$$

and we see that the identity is valid.

19. We have:

$$(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2(y + z) + 3y^2(x + z) + 3z^2(x + y) + 6xyz.$$

Hence

$$(x + y + z)^3 - x^3 - y^3 - z^3 = 3\{x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + 2xyz\} = 3\{z(x^2 + y^2 + 2xy) + z^2(x + y) + xy(x + y)\} = 3(x + y)\{z(x + y) + z^2 + xy\} = 3(x + y)(x + z)(y + z).$$

Thus,

$$(x + y + z)^3 - x^3 - y^3 - z^3 = 3(x + y)(x + z)(y + z).$$

20. We have

$$(x+y+z)^3 = x^3+y^3+z^3+3xy(x+y+z)+ \\ +3xz(x+y+z)+3yz(x+y+z)- \\ -3xyz.$$

Consequently

$$x^3+y^3+z^3-3xyz = (x+y+z)^3-3(x+y+z)\times \\ \times(xy+xz+yz) = (x+y+z)\times \\ \times(x^2+y^2+z^2-xy-xz-yz).$$

21. Put $a+b-c=x$, $b+c-a=y$, $c+a-b=$ $=z$. It is readily seen that $x+y+z=a+b+c$ and, hence, we have to simplify the following expression

$$(x+y+z)^3-x^3-y^3-z^3.$$

On the basis of Problem 19 we have

$$(x+y+z)^3-x^3-y^3-z^3 = 3(x+y)(x+z)(y+z)$$

But $x+y=2b$, $x+z=2a$, $y+z=2c$, therefore,

$$(a+b+c)^3-(a+b-c)^3-(b+c-a)^3- \\ -(c+a-b)^3 = 24abc.$$

22. On the basis of Problem 19 we have

$$x^3+y^3+z^3 = (x+y+z)^3-3(x+y)(x+z)(y+z).$$

Putting here $x=b-c$, $y=c-a$, $z=a-b$, we find

$$x+y+z=0,\quad x+y=b-a, \\ x+z=a-c,\quad y+z=c-b.$$

Hence

$$(b-c)^3+(c-a)^3+(a-b)^3 = \\ = 3(a-b)(a-c)(c-b).$$

23. Readily obtained from Problem 20. But it is possible to use the following method

$$(a+b+c)(a^2+b^2+c^2)=0.$$

since

$$a + b + c = 0.$$

Hence,

$$a^3 + b^3 + c^3 + ab(a + b) + ac(a + c) + bc(b + c) = 0.$$

But

$$a + b = -c, \ a + c = -b, \ b + c = -a.$$

Now the required identity is obvious.

24. We have

$$(a + b + c)^2 = 0,$$

$$a^2 + b^2 + c^2 = -2(ab + ac + bc).$$

Squaring both members of the latter equality, we get

$$(a^2 + b^2 + c^2)^2 = 4[a^2b^2 + a^2c^2 + b^2c^2 + 2a^2bc + 2b^2ac + 2c^2ab] = 4[a^2b^2 + a^2c^2 + b^2c^2 + 2abc(a + b + c)] = 4[a^2b^2 + a^2c^2 + b^2c^2].$$

On the other hand,

$$(a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2).$$

Hence

$$4(a^2b^2 + a^2c^2 + b^2c^2) = 2(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4).$$

Comparing it with the equality

$$4(a^2b^2 + a^2c^2 + b^2c^2) = (a^2 + b^2 + c^2)^2,$$

we get the required result.

25. Since

$$(a - b) + (b - c) + (c - a) = 0,$$

the result follows immediately from Problem 24.

26. 1° We have (see Problem 23)

$$a^3 + b^3 + c^3 = 3abc.$$

Whence

$$(a^3 + b^3 + c^3)(a^2 + b^2 + c^2) = = 3abc(a^2 + b^2 + c^2).$$

Then, transforming the left member, we obtain

$$a^5 + b^5 + c^5 + a^2b^2(a + b) + a^2c^2(a + c) + \\ + b^2c^2(b + c) = 3abc(a^2 + b^2 + c^2)$$

or

$$a^5 + b^5 + c^5 - a^2b^2c - a^2c^2b - b^2c^2a = 3abc(a^2 + b^2 + c^2).$$

Hence

$$a^5 + b^5 + c^5 - abc(ab + ac + bc) = 3abc(a^2 + b^2 + c^2).$$

But

$$-2(ab + ac + bc) = a^2 + b^2 + c^2.$$

Hence follows the final result.

2° The answer follows immediately from Problem 23 and 1°.

3° Let us write the relations

$$2(a^4 + b^4 + c^4) = (a^2 + b^2 + c^2)^2 \quad \text{(Problem 24)},$$
$$a^3 + b^3 + c^3 = 3abc \quad \text{(Problem 23)}.$$

Multiplying these equalities, we find

$$2[a^7 + b^7 + c^7 + a^3b^3(a + b) + a^3c^3(a + c) + \\ + b^3c^3(b + c)] = 3abc(a^2 + b^2 + c^2)^2.$$

Hence

$$2[a^7 + b^7 + c^7 - a^3b^3c - a^3c^3b - b^3c^3a] = \\ = 3abc(a^2 + b^2 + c^2)^2$$

or

$$2(a^7 + b^7 + c^7) - 2abc(a^2b^2 + a^2c^2 + b^2c^2) = \\ = 3abc(a^2 + b^2 + c^2)^2.$$

But

$$a^2b^2 + a^2c^2 + b^2c^2 = \frac{1}{4}(a^2 + b^2 + c^2)^2 \quad \text{(Problem 24)}.$$

Therefore

$$2(a^7 + b^7 + c^7) = \frac{7}{2}abc(a^2 + b^2 + c^2)^2.$$

Using the result of 1°, we finally get the required relation

27. For the sake of convenience let us introduce the summation symbol. And so, we put

$$\alpha_1+\alpha_2+\ldots+\alpha_n=\sum_{k=1}^{k=n}\alpha_k.$$

Using this symbol, we can now write

$$a_1b_1+a_2b_2+\ldots+a_nb_n=\sum_{k=1}^{k=n}a_kb_k=a_1b_1+\sum_{k=2}^{k=n}a_kb_k.$$

But it is obvious that

$$b_k=(b_1+b_2+\ldots+b_k)-(b_1+b_2+\ldots+b_{k-1})=$$
$$=s_k-s_{k-1},$$

therefore our sum takes the following form

$$a_1b_1+\sum_{k=2}^{k=n}a_k(s_k-s_{k-1})=a_1b_1+\sum_{k=2}^{k=n-1}a_ks_k-\sum_{k=3}^{k=n}a_ks_{k-1}+$$
$$+a_ns_n-a_2s_1=(a_1-a_2)s_1+a_ns_n+\sum_{k=2}^{k=n-1}a_ks_k-$$
$$-\sum_{k=2}^{k=n-1}a_{k+1}s_k=(a_1-a_2)s_1+\sum_{k=2}^{k=n-1}(a_k-a_{k+1})s_k+a_ns_n=$$
$$=(a_1-a_2)s_1+(a_2-a_3)s_2+\ldots+(a_{n-1}-a_n)s_{n-1}+a_ns_n.$$

28. Readily proved if we remove the brackets in the left member and use the relation

$$a_1+a_2+\ldots+a_n=\frac{n}{2}\cdot s.$$

29. Substituting into the given expression x' and y' for x and y, we find that

$$A'=A\alpha^2+2B\alpha\gamma+C\gamma^2,$$
$$C'=A\beta^2+2B\beta\delta+C\delta^2,$$
$$B'=A\alpha\beta+B(\alpha\delta+\beta\gamma)+C\gamma\delta.$$

Making up the expression $B'^2-A'C'$, we easily check the required identity.

30. We have

$$\sum_{i=1}^{i=n} p_i q_i = \sum_{i=1}^{i=n} p_i (1-p_i) = \sum_{i=1}^{i=n} p_i - \sum_{i=1}^{i=n} p_i^2 = np - \sum_{i=1}^{i=n} p_i^2,$$

since

$$np = p_1 + p_2 + \ldots + p_n.$$

Further

$$\sum_{i=1}^{i=n} p_i q_i = np - \sum_{i=1}^{i=n} (p_i - p + p)^2 =$$

$$= pn - \sum_{i=1}^{i=n} [(p_i - p)^2 + 2pp_i - p^2] = np - \sum_{i=1}^{i=n} (p_i - p)^2 -$$

$$-2p \sum_{i=1}^{i=n} p_i + np^2 = np - \sum_{i=1}^{i=n} (p_i - p)^2 - np^2.$$

But

$$np - np^2 = np(1-p) = npq.$$

Thus, we get

$$p_1q_1 + p_2q_2 + \ldots + p_nq_n = npq - (p_1 - p)^2 -$$
$$-(p_2 - p)^2 - \ldots - (p_n - p)^2.$$

31. Indeed

$$\frac{1}{1} \cdot \frac{1}{2n-1} + \frac{1}{3} \cdot \frac{1}{2n-3} + \ldots + \frac{1}{2n-1} \cdot \frac{1}{1} =$$

$$= \frac{1}{2n} \left\{ \frac{(2n-1)+1}{1 \cdot (2n-1)} + \frac{(2n-3)+3}{3(2n-3)} + \ldots + \frac{1+(2n-1)}{(2n-1) \cdot 1} \right\} =$$

$$= \frac{1}{2n} \left\{ \frac{1}{1} + \frac{1}{2n-1} + \frac{1}{3} + \frac{1}{2n-3} + \ldots + \frac{1}{2n-1} + \frac{1}{1} \right\} =$$

$$= \frac{1}{n} \left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \right).$$

32. 1° It is obvious that

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} =$$

$$= n + \left[(1-1) + \left(\frac{1}{2} - 1 \right) + \right.$$

$$\left. + \left(\frac{1}{3} - 1 \right) + \ldots + \left(\frac{1}{n} - 1 \right) \right] =$$

$$= n - \left(\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n-1}{n} \right).$$

$$2^\circ \quad s_n=\sum_{k=1}^{k=n}\frac{1}{k}, \quad ns_n=\sum_{k=1}^{k=n}\frac{n-k+k}{k}=\sum_{k=1}^{k=n}\left(\frac{n-k}{k}+1\right).$$

Hence,

$$ns_n=n+\left(\frac{n-1}{1}+\frac{n-2}{2}+\ldots+\frac{1}{n-1}\right).$$

33. Add to and subtract from the left member the following expression

$$2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right).$$

We get

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=$$

$$=\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$$

$$=\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right)+$$

$$+\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)=$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}-$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$$

34. We have

$$\left(1+\frac{1}{\alpha-1}\right)\left(1-\frac{1}{2\alpha-1}\right)\left(1+\frac{1}{3\alpha-1}\right)\ldots\times$$

$$\times\left(1+\frac{1}{(2n-1)\alpha-1}\right)\left(1-\frac{1}{2n\alpha-1}\right)=$$

$$=\frac{\alpha(2\alpha-2)\cdot 3\alpha\ \ldots\ (2n-1)\alpha(2n\alpha-2)}{(\alpha-1)(2\alpha-1)(3\alpha-1)\ \ldots\ (2n\alpha-1)}=$$

$$=\frac{1\cdot\alpha\cdot 3\cdot\alpha\cdot 5\cdot\alpha\ \ldots\ (2n-1)\cdot\alpha\cdot(2\alpha-2)(4\alpha-2)\ \ldots\ (2n\alpha-2)}{(\alpha-1)(2\alpha-1)\ldots(n\alpha-1)[(n+1)\alpha-1][(n+2)\alpha-1]\ldots[(n+n)\alpha-1]}=$$

$$=\frac{1\cdot\alpha\cdot 3\cdot\alpha\cdot 5\cdot\alpha\ \ldots\ (2n-1)\alpha\cdot(\alpha-1)(2\alpha-1)\ \ldots\ (n\alpha-1)}{[(n+1)\alpha-1]\ \ldots\ [(n+n)\alpha-1](\alpha-1)(2\alpha-1)\ \ldots\ (n\alpha-1)}\cdot 2^n=$$

$$=\frac{1\cdot\alpha\cdot 3\cdot\alpha\cdot 5\cdot\alpha\ \ldots\ (2n-1)\cdot\alpha}{[(n+1)\alpha-1]\ \ldots\ [(n+n)\alpha-1]}\cdot 2^n.$$

But

$$1\cdot3\cdot5\ \dots\ (2n-1)\cdot2^n=\frac{1\cdot2\cdot3\cdot4\cdot5\ \dots\ 2n}{2\cdot4\cdot6\ \dots\ 2n}\cdot2^n=$$

$$=\frac{1\cdot2\cdot3\cdot4\cdot5\ \dots\ 2n}{1\cdot2\cdot3\ \dots\ n}=(n+1)(n+2)\ \dots\ 2n,$$

wherefrom we obtain the required identity.

35. Let $a \leqslant x < a+1$, where a is an integer. Subdivide the interval between a and $a+1$ into n parts. Then x will lie in one of these subintervals, i.e. we can find a whole number p $(0 \leqslant p < n-1)$ such that

$$a+\frac{p}{n}\leqslant x<a+\frac{p+1}{n}.$$

Therefore

$$a+\frac{p+1}{n}\leqslant x+\frac{1}{n}<a+\frac{p+2}{n},$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$a+1-\frac{1}{n}\leqslant x+\frac{n-p-1}{n}<a+1,$$

$$a+1\leqslant x+\frac{n-p}{n}<a+1+\frac{1}{n},$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$a+\frac{p+n-1}{n}\leqslant x+\frac{n-1}{n}<a+\frac{p+n}{n}.$$

Hence

$$[x]=\left[x+\frac{1}{n}\right]=\dots=\left[x+\frac{n-p-1}{n}\right]=a$$

$$\left[x+\frac{n-p}{n}\right]=\dots=\left[x+\frac{n-1}{n}\right]=a+1.$$

Consequently

$$[x]+\left[x+\frac{1}{n}\right]+\dots+\left[x+\frac{n-1}{n}\right]=$$

$$=(n-p)a+p(a+1)=an+p.$$

On the other hand, from the inequality

$$a+\frac{p}{n}\leqslant x<a+\frac{p+1}{n}$$

we get

$$an + p \leqslant nx < an + p + 1,$$

hence,

$$[nx] = an + p,$$

and the formula is proved.

36. We have

$$\cos(a+b)\cos(a-b) = $$
$$= [\cos a \cos b - \sin a \sin b] \times$$
$$\times [\cos a \cos b + \sin a \sin b] = \cos^2 a \cos^2 b -$$
$$- \sin^2 a \sin^2 b = \cos^2 a\,(1 - \sin^2 b) -$$
$$- (1 - \cos^2 a)\sin^2 b = \cos^2 a - \sin^2 b.$$

37. Expanding the bracketed expressions in the left members, we easily prove the equalities.

38. We have

$$(1-\sin a)(1-\sin b)(1-\sin c) =$$
$$= \frac{(1-\sin^2 a)(1-\sin^2 b)(1-\sin^2 c)}{(1+\sin a)(1+\sin b)(1+\sin c)} =$$
$$= \frac{\cos^2 a \cos^2 b \cos^2 c}{\cos a \cos b \cos c} = \cos a \cos b \cos c.$$

39. Multiplying both members of the given equality by

$$(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma),$$

we get

$$[(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma)]^2 =$$
$$= \sin^2\alpha \sin^2\beta \sin^2\gamma.$$

40. Using the formula

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)],$$

we get

$$2\cos(\alpha+\beta)\sin(\alpha-\beta) = \sin 2\alpha - \sin 2\beta,$$
$$2\cos(\beta+\gamma)\sin(\beta-\gamma) = \sin 2\beta - \sin 2\gamma$$

and so on. Hence follows the identity.

41. Using the formula

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)],$$

we get the identity

$$(\cos 2b - \cos 2a)(\cos 2d - \cos 2c) + \\ + (\cos 2b - \cos 2c)(\cos 2a - \cos 2d) + \\ + (\cos 2b - \cos 2d)(\cos 2c - \cos 2a) = 0.$$

Let $\cos 2b = \alpha$, $\cos 2a = \beta$, $\cos 2d = \gamma$, $\cos 2c = \delta$, then

$$(\alpha-\beta)(\gamma-\delta) + (\alpha-\delta)(\beta-\gamma) + (\alpha-\gamma)(\delta-\beta) = \\ = (\alpha-\beta)(\gamma-\delta) + (\alpha-\gamma+\gamma-\delta)(\beta-\gamma) + \\ + (\alpha-\gamma)(\delta-\beta) = (\alpha-\beta)(\gamma-\delta) + \\ + (\alpha-\gamma)(\beta-\gamma) + (\gamma-\delta)(\beta-\gamma) + \\ + (\alpha-\gamma)(\delta-\beta) = 0.$$

But $(\alpha-\beta)(\gamma-\delta) + (\gamma-\delta)(\beta-\gamma) = (\gamma-\delta)(\alpha-\gamma)$ and $(\alpha-\gamma)(\beta-\gamma) + (\alpha-\gamma)\cdot(\delta-\beta) = (\alpha-\gamma)(\delta-\gamma)$; hence the required sum is equal to $(\alpha-\gamma)(\gamma-\delta) + (\alpha-\gamma)(\delta-\gamma) = 0$.

42. 1° Summing the first two cosines, we get $2\cos\gamma\cos(\beta-\alpha)$; the sum of the second two cosines yields $2\cos(\alpha+\beta)\cos\gamma$. The further check is obvious.

2° Analogous to 1°.

43. We have

$$\sin\left(A+\frac{B}{4}\right) + \cos\left(A+\frac{B}{4}\right) = \sin\left(A+\frac{B}{4}\right) + \\ + \sin\left(\frac{\pi}{2}-A-\frac{B}{4}\right) = 2\sin\frac{\pi}{4}\cos\left(\frac{\pi}{4}-A-\frac{B}{4}\right).$$

With the aid of a circular permutation we obtain (denoting the transformed sum by S)

$$\frac{S}{\sqrt{2}} = \cos\left(\frac{\pi}{4}-A-\frac{B}{4}\right) + \cos\left(\frac{\pi}{4}-B-\frac{C}{4}\right) + \\ + \cos\left(\frac{\pi}{4}-C-\frac{A}{4}\right) = 2\cos\left(\frac{\pi}{4}-\frac{A+B}{2}-\frac{B+C}{8}\right)\times \\ \times\cos\left(\frac{A-B}{2}+\frac{B+C}{8}\right) + \sin\left(\frac{\pi}{4}+C+\frac{A}{4}\right).$$

Making use of the relation $A+B+C=\pi$, we can show that

$$\cos\left(\frac{\pi}{4}-\frac{A+B}{2}-\frac{B+C}{8}\right)=\sin\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right).$$

Therefore we have

$$\frac{S}{\sqrt{2}}=2\sin\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right)\cos\left(\frac{A-B}{2}+\frac{B-C}{8}\right)+$$
$$+2\sin\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right)\cos\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right)=$$
$$=2\sin\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right)\left[\cos\left(\frac{A-B}{2}+\frac{B-C}{8}\right)+\right.$$
$$\left.+\cos\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right)\right]=4\sin\left(\frac{\pi}{8}+\frac{A}{2}+\frac{B}{8}\right)\times$$
$$\times\sin\left(\frac{\pi}{8}+\frac{B}{2}+\frac{C}{8}\right)\sin\left(\frac{\pi}{8}+\frac{C}{2}+\frac{A}{8}\right).$$

44. Carrying out some transformations analogous to the previous ones, we obtain the following result

$$\sin\frac{A}{4}+\sin\frac{B}{4}+\sin\frac{C}{4}+\cos\frac{A}{4}+\cos\frac{B}{4}+\cos\frac{C}{4}=$$
$$=4\sqrt{2}\cos\left(\frac{\pi}{8}+\frac{C}{8}\right)\cos\left(\frac{\pi}{8}+\frac{B}{8}\right)\cos\left(\frac{\pi}{8}+\frac{A}{8}\right).$$

45. We have

$$\sin 2a = 2\sin a\cos a,$$
$$\sin 4a = 2\sin 2a\cos 2a,$$
$$\sin 8a = 2\sin 4a\cos 4a,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\sin 2^n a = 2\sin 2^{n-1}a\cos 2^{n-1}a.$$

Multiplying term by term and dividing both members by the product

$$\sin 2a\sin 4a\ldots\sin 2^{n-1}a,$$

we get

$$\sin 2^n a = 2^n\sin a\cos a\cos 2a\ldots\cos 2^{n-1}a,$$

whence

$$\cos a \cos 2a \ \ldots \ \cos 2^{n-1}a = \frac{\sin 2^n a}{2^n \sin a}.$$

46. We have

$$\sin\frac{2\pi}{15} = 2\sin\frac{\pi}{15}\cos\frac{\pi}{15}, \quad \sin\frac{4\pi}{15} = 2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15},$$

$$\sin\frac{8\pi}{15} = 2\sin\frac{4\pi}{15}\cos\frac{4\pi}{15}, \quad \sin\frac{16\pi}{15} = 2\sin\frac{8\pi}{15}\cos\frac{8\pi}{15}.$$

Multiplying the equalities and noting that $\sin\frac{16\pi}{15} = -\sin\frac{\pi}{15}$, $\cos\frac{8\pi}{15} = -\cos\frac{7\pi}{15}$, we find

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^4}.$$

Further

$$\cos\frac{5\pi}{15} = \frac{1}{2}$$

and

$$\sin\frac{6\pi}{15} = 2\sin\frac{3\pi}{15}\cos\frac{3\pi}{15}, \quad \sin\frac{12\pi}{15} = 2\sin\frac{6\pi}{15}\cos\frac{6\pi}{15}.$$

Hence

$$\cos\frac{3\pi}{15}\cdot\cos\frac{6\pi}{15} = \frac{1}{2^2}.$$

The rest is obvious.

47. We have

$$\frac{\tan(A+B)}{\tan A} = \frac{\sin(A+B)\cos A}{\cos(A+B)\sin A} = \frac{\sin(2A+B)+\sin B}{\sin(2A+B)-\sin B} = \frac{3}{2}.$$

48. From the given relations we get

$$\sin 2B = \frac{3}{2}\sin 2A,$$

$$3\sin^2 A = 1 - 2\sin^2 B = \cos 2B,$$

hence

$$\cos(A+2B) = \cos A\cos 2B - \sin A\sin 2B =$$
$$= \cos A\cdot 3\sin^2 A - \frac{3}{2}\sin A\sin 2A = 0.$$

49. We have

$$2 \cos a \cos \varphi = \cos (a + \varphi) + \cos (a - \varphi).$$

Consequently the expression under consideration is equal to

$$\cos^2 \varphi + \cos^2 (a + \varphi) - [\cos^2 (a + \varphi) + \\ + \cos (a + \varphi) \cos (a - \varphi)] = \cos^2 \varphi - \\ - \cos^2 a \cos^2 \varphi + \sin^2 a \sin^2 \varphi = \sin^2 a.$$

50. We have, for instance,

$$a^2 + a'^2 + a''^2 = \cos^2 \varphi \cos^2 \psi + \sin^2 \varphi \sin^2 \psi \cos^2 \delta + \\ + \cos^2 \varphi \sin^2 \psi + \sin^2 \varphi \cos^2 \psi \cos^2 \delta + \sin^2 \varphi \sin^2 \delta$$

(the doubled products in the first two squares are cancelled out). Hence

$$a^2 + a'^2 + a''^2 = (\cos^2 \varphi \cos^2 \psi + \cos^2 \varphi \sin^2 \psi) + \\ + (\sin^2 \varphi \sin^2 \psi \cos^2 \delta + \sin^2 \varphi \cos^2 \psi \cos^2 \delta) + \\ + \sin^2 \varphi \sin^2 \delta = \cos^2 \varphi + \\ + (\sin^2 \varphi \cos^2 \delta + \sin^2 \varphi \sin^2 \delta) = 1.$$

The remaining equalities are proved similarly.

SOLUTIONS TO SECTION 2

1. Rewrite the identity in the following way

$$q^3 + q^3 \frac{(2p^3 - q^3)^3}{(p^3 + q^3)^3} = p^3 - p^3 \frac{(p^3 - 2q^3)^3}{(p^3 + q^3)^3} .$$

It is evident that the right member can be obtained from the left one by permuting p and q. Let us reduce the left member to such a form, wherefrom it would be seen that after the permutation its value remains unchanged. Then the validity of the identity will become clear.

We have

$$\frac{q^3}{(p^3 + q^3)^3} \{(p^3 + q^3)^3 + (2p^3 - q^3)^3\} = \frac{9p^3q^3}{(p^3 + q^3)^3} (p^6 + q^6 - p^6q^6).$$

2. We have

$$\frac{p^3+q^3}{(p+q)^3\,p^3q^3}+\frac{3}{(p+q)^4}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)+\frac{6(p+q)}{(p+q)^5\,pq}=$$

$$=\frac{p^2-pq-q^2}{(p+q)^2\,p^3q^3}+\frac{3}{(p+q)^4}\left(\frac{1}{p^2}+\frac{1}{q^2}+\frac{2}{pq}\right)=$$

$$=\frac{p^2-pq+q^2}{(p+q)^2\,p^3q^3}+\frac{3}{(p+q)^4}\left(\frac{1}{p}+\frac{1}{q}\right)^2=$$

$$=\frac{p^2-pq+q^2}{(p+q)^2\,p^3q^3}+\frac{3}{(p+q)^2\,p^2q^2}=\frac{1}{(p+q)^2\,p^3q^3}\times$$

$$\times\{p^2-pq+q^2+3pq\}=\frac{1}{p^3q^3}.$$

3. Grouping the last two terms of the sum, we get

$$\frac{2}{(p+q)^4}\,\frac{q^3-p^3}{p^3q^3}+\frac{2}{(p+q)^4}\,\frac{q-p}{p^2q^2}=$$

$$=\frac{2(q-p)}{(p+q)^4\,p^3q^3}\,(p^2+q^2+2pq)=\frac{2(q-p)}{(p+q)^2\,p^3q^3}.$$

Adding now the first term, we find

$$\frac{1}{(p+q)^3}\,\frac{q^4-p^4}{p^4q^4}+\frac{2(q-p)}{(p+q)^2\,p^3q^3}=\frac{q-p}{p^4q^4}.$$

4. We have to prove that

$$\frac{1+x}{1-x}\cdot\frac{1+y}{1-y}\cdot\frac{1+z}{1-z}=1.$$

Replacing x by its expression, we find $\frac{1+x}{1-x}=\frac{a}{b}$. Since y and z are obtained from x by means of a circular permutation of the letters a, b, c, we have

$$\frac{1+y}{1-y}=\frac{b}{c},$$

$$\frac{1+z}{1-z}=\frac{c}{a}.$$

Hence, the required identity is obvious.

5. We have

$$\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}.$$

But if $\frac{A}{B}=\frac{C}{D}$, then $\frac{A+B}{A-B}=\frac{C+D}{C-D}$, and conversely if there exists the second of these equalities, then the first one exists as well. Reasoning in the same way (putting $A=a+b+c+d$, $B=a+b-c-d$, $C=a-b+c-d$, $D=a-b-c+d$, we find

$$\frac{a+b}{c+d}=\frac{a-b}{c-d} \text{ or } \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

Hence

$$\frac{a}{b}=\frac{c}{d} \text{ or } \frac{a}{c}=\frac{b}{d}.$$

6. The denominator has the form

$$\begin{aligned} &bcy^2+bcz^2-2bcyz+acz^2+acx^2-2acxz+abx^2+\\ &\quad+aby^2-2abxy=c(ax^2+by^2)+b(ax^2+cz^2)+\\ &\quad+a(cz^2+by^2)-2bcyz-2acxz-2abxy=\\ &\quad=(a+b+c)(ax^2+by^2+cz^2)-c^2z^2-b^2y^2-\\ &\quad-a^2x^2-2bcyz-2acxz-2abxy=(a+b+c)\times\\ &\quad\times(ax^2+by^2+cz^2)-(ax+by+cz)^2. \end{aligned}$$

Since, by hypothesis, $ax+by+cz=0$, the denominator turns out to be equal to

$$(a+b+c)(ax^2+by^2+cz^2),$$

and our fraction is equal to

$$\frac{1}{a+b+c}.$$

7. Reduce to a common denominator the expression on the left. The numerator of the fraction obtained will be equal to

$$\begin{aligned} &x^2y^2z^2(a^2-b^2)+b^2(x^2-a^2)(y^2-a^2)(z^2-a^2)-\\ &\qquad-a^2(x^2-b^2)(y^2-b^2)(z^2-b^2). \end{aligned}$$

It is obvious that

$$\begin{aligned} &(a^2-x^2)(a^2-y^2)(a^2-z^2)=\\ &\quad=a^6-(x^2+y^2+z^2)a^4+(x^2y^2+x^2z^2+y^2z^2)a^2-\\ &\quad-x^2y^2z^2. \end{aligned}$$

Hence

$$(b^2 - x^2)(b^2 - y^2)(b^2 - z^2) = \\ = b^6 - (x^2 + y^2 + z^2)\, b^4 + \\ + (x^2y^2 + x^2z^2 + y^2z^2)\, b^2 - x^2y^2z^2.$$

Substituting these expressions into the numerator and performing all the necessary transformations, we obtain the required value of the fraction.

8. $S_0 = \frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}.$

Reducing the fractions to a common denominator, we have

$$S_0 = \frac{1}{(a-b)(a-c)(b-c)} \{(b-c) - (a-c) + (a-b)\} = 0,$$

$$S_1 = \frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)} = \\ = \frac{1}{(a-b)(a-c)(b-c)} \{a(b-c) - b(a-c) + c(a-b)\} = 0,$$

$$S_2 = \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = \\ = \frac{1}{(a-b)(a-c)(b-c)} \{a^2(b-c) - b^2(a-c) + c^2(a-b)\}.$$

Consider the numerator.

We have

$$a^2(b-c) - b^2(a-c) + c^2(a-b) = \\ = ab(a-b) - c(a^2 - b^2) + c^2(a-b) = \\ = (a-b)(ab - ca - cb + c^2) = \\ = (a-b)[a(b-c) - c(b-c)] = \\ = (a-b)(b-c)(a-c),$$

wherefrom it follows that $S_2 = 1$. S_3, S_4 and S_5 can be computed analogously, but we shall proceed here in a somewhat different way.

It is easily seen that there exists the following identity

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + \\ + (ab + ac + bc)\, x - abc.$$

Putting, $x = a$, $x = b$ and $x = c$, in turn, we get the following equalities

$$a^3 - (a + b + c)\,a^2 + (ab + ac + bc)\,a - abc = 0,$$
$$b^3 - (a + b + c)\,b^2 + (ab + ac + bc)\,b - abc = 0,$$
$$c^3 - (a + b + c)\,c^2 + (ab + ac + bc)\,c - abc = 0.$$

Further, divide the first of them by $(a - b)(a - c)$, the second by $(b - c)(b - a)$ and the third by $(c - a) \times \times (c - b)$, and add them term by term. Then

$$S_3 - (a + b + c)\,S_2 + (ab + ac + bc)\,S_1 - abc\,S_0 = 0.$$

But since it is known that $S_0 = S_1 = 0$, $S_2 = 1$, we have: $S_3 = a + b + c$.

To compute S_4 let us take the preceding identity and multiply its members by x. We obtain

$$x(x - a)(x - b)(x - c) = x^4 - (a + b + c)\,x^3 + \\ + (ab + ac + bc)\,x^2 - abcx.$$

Proceeding analogously, we find:

$$S_4 - (a + b + c)\,S_3 + (ab + ac + bc)\,S_2 - abc\,S_1 = 0.$$

Hence

$$S_4 = (a + b + c)\,S_3 - (ab + ac + bc)\,S_2 = \\ = (a + b + c)^2 - ab - ac - bc = \\ = a^2 + b^2 + c^2 + ab + ac + bc.$$

Likewise, for computing S_5 (multiplying the original identity by x^2), we find

$$S_5 - (a + b + c)\,S_4 + (ab + ac + bc)\,S_3 - abc\,S_2 = 0.$$

Consequently

$$S_5 = (a + b + c)(a^2 + b^2 + c^2 + ab + ac + bc) - \\ - (ab + ac + bc)(a + b + c) + abc = \\ = (a + b + c)(a^2 + b^2 + c^2) + abc = \\ = a^3 + b^3 + c^3 + a^2b + a^2c + b^2a + b^2c + \\ + c^2a + c^2b + abc.$$

9. This problem is solved analogously to the preceding one. Namely, the equalities $S_0 = S_1 = S_2 = 0$, $S_3 = 1$ are established by a direct check; and to compute S_4 we may resort to the following identity

$$(x - a)(x - b)(x - c)(x - d) = \\ = x^4 - (a + b + c + d)x^3 + \\ + (ab + ac + ad + bc + bd + dc)x^2 - \\ - (abc + abd + acd + bcd)x + abcd$$

Hence

$$S_4 = (a + b + c + d)S_3 = a + b + c + d.$$

10. Put as before

$$S_m = \frac{a^m}{(a-b)(a-c)} + \frac{b^m}{(b-a)(b-c)} + \frac{c^m}{(c-a)(c-b)}.$$

Let us take the first term of our sum σ_m and transform it

$$a^m \frac{(a+b)(a+c)}{(a-b)(a-c)} = \frac{(a+b+c)a^{m+1} + a^{m-1}.abc}{(a-b)(a-c)}.$$

Making use of a circular permutation, we get similar expressions for the second and third terms of σ_m. Adding now all these terms, we find: $\sigma_m = (a + b + c)S_{m+1} + abc\,S_{m-1}$. Hence (after some transformations)

$$\sigma_1 = (a + b + c)S_2 + abc\,S_0 = a + b + c \quad (S_2 = 1,\ S_0 = 0),$$

$$\sigma_2 = (a + b + c)S_3 + abc\,S_1 = (a + b + c)^2,$$

since $S_3 = a + b + c$, $S_1 = 0$,

$$\sigma_3 = (a + b + c)S_4 + abc\,S_2 = \\ = (a + b + c)(a^2 + b^2 + c^2 + ab + ac + bc) + abc,$$

$$\sigma_4 = (a + b + c)S_5 + abc\,S_3 = \\ = (a + b + c)[(a + b + c)(a^2 + b^2 + c^2) + 2abc].$$

11. Transform the left member of our identity in the following way

$$abc\left\{\frac{(a-\alpha)(a-\beta)(a-\gamma)}{(a-0)(a-b)(a-c)}+\frac{(b-\alpha)(b-\beta)(b-\gamma)}{(b-0)(b-a)(b-c)}+\right.$$
$$\left.+\frac{(c-\alpha)(c-\beta)(c-\gamma)}{(c-0)(c-a)(c-b)}+\frac{(0-\alpha)(0-\beta)(0-\gamma)}{(0-c)(0-a)(c-b)}-\frac{\alpha\beta\gamma}{abc}\right\}.$$

Consider the first four terms of the sum in braces. Expanding the numerator of the first term in powers of a, we get

$$a^3-(\alpha+\beta+\gamma)a^2+(\alpha\beta+\alpha\gamma+\beta\gamma)a-\alpha\beta\gamma.$$

Performing an analogous operation with the remaining three terms and adding them, we find that the sum of the first four terms is equal to

$$S_3-(\alpha+\beta+\gamma)S_2+(\alpha\beta+\alpha\gamma+\beta\gamma)S_1-\alpha\beta\gamma S_0,$$

where S_k is the known sum (see Problem 9, where it is necessary to put $d=0$). Proceeding from the results of this problem, we find that the sum of the first four terms under consideration is equal to unity, and, consequently, the sought-for expression takes the form

$$abc\left\{1-\frac{\alpha\beta\gamma}{abc}\right\}=abc-\alpha\beta\gamma.$$

12. Consider the following sum:

$$S_4=\frac{\alpha^4}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^4}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+$$
$$+\frac{\gamma^4}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

From Problem 9 we have: $S_4=\alpha+\beta+\gamma+\delta$. Put $\alpha=$ $=abc$, $\beta=abd$, $\gamma=acd$, $\delta=bcd$. Then

$$\frac{\alpha^4}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}=\frac{a^4b^4c^4}{(abc-abd)(abc-acd)(abc-bcd)}=$$
$$=\frac{a^2b^2c^2}{(c-d)(b-d)(a-d)}.$$

Using a circular permutation, we get analogous expression for the remaining three terms. Thus, the given identity is proved.

13. 1° Transform one of the terms in the following way:

$$\frac{1}{a(a-b)(a-c)}=\frac{1}{a}\,\frac{\frac{1}{ab}\cdot\frac{1}{ac}}{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{a}\right)}=$$

$$=\frac{1}{abc}\,\frac{\left(\frac{1}{a}\right)^2}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)}.$$

Then the required sum is equal to

$$\frac{1}{abc}\left\{\frac{\left(\frac{1}{a}\right)^2}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)}+\frac{\left(\frac{1}{b}\right)^2}{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{c}\right)}+\right.$$

$$\left.+\frac{\left(\frac{1}{c}\right)^2}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right)}\right\}=\frac{1}{abc}S_2.$$

But (see Problem 8) $S_2 = 1$, and, hence, we get:

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}=\frac{1}{abc}.$$

However, this result can be obtained in a somewhat different way. Let us consider the four quantities: a, b, c and 0, and form S_0 for them.

We then have

$$S_0=\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}+$$

$$+\frac{1}{(0-a)(0-b)(0-c)}=0,$$

since $S_0 = 0$. Hence we get the previous result.

2° Likewise the sum can be transformed as

$$\frac{1}{abc}\left\{\frac{\left(\frac{1}{a}\right)^3}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)}+\frac{\left(\frac{1}{b}\right)^3}{\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{c}\right)}+\right.$$

$$\left.+\frac{\left(\frac{1}{c}\right)^3}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right)}\right\}=\frac{1}{abc}S_3=\frac{1}{abc}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

And so

$$\frac{1}{a^2(a-b)(a-c)}+\frac{1}{b^2(b-a)(b-c)}+\frac{1}{c^2(c-a)(c-b)}=$$
$$=\frac{ab+ac+bc}{a^2b^2c^2}.$$

A similar method can be applied when computing other sums of the form

$$\frac{1}{a^k(a-b)(a-c)}+\frac{1}{b^k(b-a)(b-c)}+\frac{1}{c^k(c-a)(c-b)}.$$

14. We have

$$\frac{a^k}{(a-b)(a-c)(a-x)}+\frac{b^k}{(b-a)(b-c)(b-x)}+$$
$$+\frac{c^k}{(c-a)(c-b)(c-x)}+\frac{x^k}{(x-a)(x-b)(x-c)}=0$$

at $k=1$ and at $k=2$ (Problem 9).

Hence

$$\frac{a^k}{(a-b)(a-c)(x-a)}+\frac{b^k}{(b-a)(b-c)(x-b)}+$$
$$+\frac{c^k}{(c-a)(c-b)(x-c)}=\frac{x^k}{(x-a)(x-b)(x-c)} \quad (k=1,\ 2).$$

15. We have

$$\frac{b+c+d}{(b-a)(c-a)(d-a)(x-a)}=\frac{(a+b+c+d-x)+(x-a)}{(b-a)(c-a)(d-a)(x-a)}=$$
$$=(a+b+c+d-x)\frac{1}{(b-a)(c-a)(d-a)(x-a)}+$$
$$+\frac{1}{(b-a)(c-a)(d-a)}.$$

Applying a circular permutation to the letters a, b, c, d and adding the expressions thus obtained, we find that the sum in the left member is equal to

$$(a+b+c+d-x)\left\{\frac{1}{(a-b)(a-c)(a-d)(a-x)}+\right.$$
$$+\frac{1}{(b-a)(b-c)(b-d)(b-x)}+\frac{1}{(c-a)(c-b)(c-d)(c-x)}+$$
$$\left.+\frac{1}{(d-a)(d-b)(d-c)(d-x)}\right\}.$$

since the second sum equals zero.

It remains only to make sure that

$$\frac{1}{(a-b)(a-c)(a-d)(a-x)}+\frac{1}{(b-a)(b-c)(b-d)(b-x)}+$$
$$+\frac{1}{(c-a)(c-b)(c-d)(c-x)}+\frac{1}{(d-a)(d-b)(d-c)(d-x)}+$$
$$+\frac{1}{(x-a)(x-b)(x-c)(x-d)}=0.$$

It is possible to reduce these fractions to a common denominator and, on performing necessary transformations in the numerator, to obtain zero. But we can, however, proceed in a different way.

Multiplying the left member by $(a-x)(b-x)(c-x)\times$ $\times(d-x)$, we get

$$\frac{1}{(a-b)(a-c)(a-d)}(b-x)(c-x)(d-x)+$$
$$+\frac{1}{(b-a)(b-c)(b-d)}(a-x)(c-x)(d-x)+$$
$$+\frac{1}{(c-a)(c-b)(c-d)}(a-x)(b-x)(d-x)+$$
$$+\frac{1}{(d-a)(d-b)(d-c)}(a-x)(b-x)(c-x)+1.$$

It is obvious that we deal with a third-degree polynomial in x. It is required to prove that it is identically equal to zero. For this purpose it is sufficient to show (see the beginning of the section) that it becomes zero at four different particular values of x. Replacing x successively by a, b, c, d, we make sure that our polynomial vanishes at these four values of x, and, consequently, it is identically equal to zero.

16. Transposing x^2 to the left, we get there a second-degree trinomial in x. To prove that it identically equals zero it suffices to show that it becomes zero at three different values of x. Putting $x=a, b, c$, we make sure that the identity is valid.

17. Solved analogously to the preceding problem. However, Problem 16, as well as this one, can be solved by making use of the quantities S_k (see Problem 8 and the following ones).

18. Put

$$\frac{a-b}{c}=x, \quad \frac{b-c}{a}=y, \quad \frac{c-a}{b}=z.$$

The left member of our equality takes the form

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=3+\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z}.$$

Consider the fraction $\frac{y+z}{x}$. We have

$$\frac{y+z}{x}=\left(\frac{b-c}{a}+\frac{c-a}{b}\right)\cdot\frac{c}{a-b}=\frac{c}{a-b}\cdot\frac{b^2-bc+ac-a^2}{ab}=$$

$$=\frac{c}{a-b}\cdot\frac{b^2-a^2-c(b-a)}{ab}=\frac{c}{ab}(-a-b+c)=$$

$$=\frac{c}{ab}(-a-b-c+2c)=\frac{2c^2}{ab},$$

since $a+b+c=0$. Using a circular permutation, we find

$$\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z}=\frac{2c^2}{ab}+\frac{2a^2}{bc}+\frac{2b^2}{ac}=\frac{2}{abc}(a^3+b^3+c^3).$$

But if $a+b+c=0$, then $a^3+b^3+c^3=3abc$ (see Problem 23, Sec. 1). Consequently

$$\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z}=6,$$

and the equality is solved.

19. Miltiplying the given expression by $(a+b)(b+c)\times\times(c+a)$, we get $(a-b)(a+c)(b+c)+(a+c)\times\times(a+b)(b-c)+(a+b)(c-a)(b+c)++(a-b)(c-a)(b-c)$.

This expression is a second-degree trinomial in a which becomes zero at $a=b$, $a=c$ and $a=0$ and, consequently, is identically equal to zero, i.e.

$$\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}+\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}=0.$$

We assume here $b\neq c$. If $b=c$, then it is easy to make sure directly that the identity holds true.

20. We have

$$\frac{b-c}{(a-b)(a-c)}=\frac{(b-a)+(a-c)}{(a-b)(a-c)}=\frac{1}{a-b}-\frac{1}{a-c}.$$

Treating the remaining two terms in a similar way, we arrive at the proposed identity.

21. *Answer.* 0. Solved analogously to Problem 19.

22. It is required to prove that

$$\frac{d^m (a-b)(b-c)+b^m (a-d)(c-d)}{c^m (a-b)(a-d)+a^m (b-c)(c-d)}-\frac{b-d}{a-c}=0.$$

Reducing to a common denominator, let us prove that the numerator equals zero. However, if the numerator is divided by the product $(a-b)(a-c)(a-d)(b-c)(b-d)\times(c-d)$, we get the following expression

$$\frac{a^m}{(a-b)(a-c)(a-d)}+\frac{b^m}{(b-a)(b-c)(b-d)}+$$
$$+\frac{c^m}{(c-a)(c-b)(c-d)}+\frac{d^m}{(d-a)(d-o)(d-c)}.$$

At $m=1, 2$ this expression is equal to zero (see Problem 9).

23. Let us first prove that

$$1-\frac{x}{\alpha_1}+\frac{x(x-\alpha_1)}{\alpha_1\alpha_2}-\frac{x(x-\alpha_1)(x-\alpha_2)}{\alpha_1\alpha_2\alpha_3}+\ldots+$$
$$+(-1)^n\frac{x(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})}{\alpha_1\alpha_2\ldots\alpha_n}=$$
$$=(-1)^n\frac{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)}{\alpha_1\alpha_2\ldots\alpha_n}. \qquad (*)$$

Likewise, it is evident that the second bracketed expression is equal to

$$\frac{(x+\alpha_1)(x+\alpha_2)\ldots(x+\alpha_n)}{\alpha_1\alpha_2\ldots\alpha_n}.$$

And the product of the bracketed expressions yields

$$(-1)^n\frac{(x^2-\alpha_1^2)(x^2-\alpha_2^2)\ldots(x^2-\alpha_n^2)}{\alpha_1^2\alpha_2^2\ldots\alpha_n^2}.$$

Replacing here x by x^2 and α_i by α_i^2 and applying the equality $(*)$ in a reverse order, we get the required identity.

24. Given

$$\left(\frac{b^2+c^2-a^2}{2bc}-1\right)+\left(\frac{c^2+a^2-b^2}{2ac}-1\right)+$$
$$+\left(\frac{a^2+b^2-c^2}{2ab}+1\right)=0. \qquad (*)$$

The first bracketed expression is equal to

$$\frac{(b-c)^2-a^2}{2bc}=\frac{(b-c-a)(b-c+a)}{2bc},$$

the second to

$$\frac{(a-c)^2-b^2}{2ac}=\frac{(a-c-b)(a-c+b)}{2ac}.$$

Likewise, the third one takes the form

$$\frac{(a+b)^2-c^2}{2ab}=\frac{(a+b+c)(a+b-c)}{2ab}.$$

Consider the sum of these expressions

$$-\frac{(a+b-c)(a+c-b)}{2bc}-\frac{(a+b-c)(c+b-a)}{2ac}+$$
$$+\frac{(a+b-c)(a+b+c)}{2ab}=$$
$$=\frac{a+b-c}{2abc}\{c(a+b+c)-b(c+b-a)-a(a+c-b)\}=$$
$$=\frac{(a+b-c)(c+a-b)(c-a+b)}{2abc}.$$

Thus, we are given that

$$\frac{(a+b-c)(a+c-b)(c+b-a)}{2abc}=0,$$

wherefrom follows that at least one of the factors in the numerator equals zero. Suppose $a+b-c=0$; then all the three bracketed expressions in the equality (*) are equal to zero, and, consequently two of the given fractions are equal to 1, while the third one to -1. The remaining two possibilities yield the same result.

25. Reducing the original equality to a common denominator and cancelling it out, we get (after some transformations)

$$(a+b)(a+c)(b+c)=0. \qquad (1)$$

But the second equality (which is to be proved) can also be reduced to the form

$$(a^n+b^n)(a^n+c^n)(b^n+c^n)=0. \qquad (2)$$

It is quite obvious, that with an odd n equality (2) follows from (1), since if, for instance, $a+b=0$, then $a=-b$ and $a^n+b^n=a^n+(-a)^n=a^n-a^n=0$.

26. Rewrite the given proportion in the following way

$$\frac{(bz+cy)\,yz}{-ax+by+cz}=\frac{(cx+az)\,xz}{ax-by+cz}=\frac{(ay+bx)\,xy}{ax+by-cz}.$$

But from the proportion $\frac{A}{B}=\frac{C}{D}=\frac{E}{F}$ follows $\frac{A+C}{B+D}=\frac{C+E}{D+F}=\frac{A+E}{B+F}$ (it is easy to check, putting $\frac{A}{B}=\frac{C}{D}=\frac{E}{F}=\lambda$ and expressing A, C and E in terms of λ, B, D, F).

Therefore we have

$$\frac{c(x^2+y^2)+z(ax+by)}{c}=\frac{a(z^2+y^2)+x(by+cz)}{a}=\frac{b(x^2+z^2)+y(cz+ax)}{b}.$$

Subtracting $x^2+y^2+z^2$ from each term of this equality, we get

$$\frac{z(ax+by-cz)}{c}=\frac{x(by+cz-ax)}{a}=\frac{y(cz+ax-by)}{b}.$$

Take the original equalities

$$\frac{ay+bx}{z(ax+by-cz)}=\frac{bz+cy}{x(-ax+by+cz)}=\frac{cx+az}{y(ax-by+cz)}.$$

Multiplying these equalities, we find

$$\frac{ay+bx}{c}=\frac{bz+cy}{a}=\frac{cx+az}{b}.$$

Hence

$$c=(ay+bx)\mu,$$
$$b=(cx+az)\mu,$$
$$a=(bz+cy)\mu.$$

Multiplying the first of these equalities by c, the second by b and the third by a, and forming the expression $b^2+c^2-a^2$, we find $b^2+c^2-a^2=2\mu bcx$.

Analogously, we get

$$c^2+a^2-b^2=2\mu cay,\quad a^2+b^2-c^2=2\mu abz.$$

Hence, finally

$$\frac{x}{a(b^2+c^2-a^2)}=\frac{y}{b(a^2+c^2-b^2)}=\frac{z}{c(a^2+b^2-c^2)}.$$

27. Since $a+b+c=0$, we may write

$$(a+b+c)(a\alpha+b\beta+c\gamma)=0.$$

Expanding the expression in the left member, we find

$$a^2\alpha+b^2\beta+c^2\gamma+ab(\alpha+\beta)+ac(\alpha+\gamma)+$$
$$+cb(\beta+\gamma)=0.$$

But $\alpha+\beta=-\gamma$, $\alpha+\gamma=-\beta$, $\beta+\gamma=-\alpha$, therefore $a^2\alpha+b^2\beta+c^2\gamma-ab\gamma-ac\beta-cb\alpha=0$, or $a^2\alpha+b^2\beta+$ $+c^2\gamma-abc\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}\right)=0$, and since $\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0$ (by hypothesis), we have: $a^2\alpha+b^2\beta+c^2\gamma=0$.

28. From the equalities

$$(b^2+c^2-a^2)x=(c^2+a^2-b^2)y=(a^2+b^2-c^2)z$$

follows

$$\frac{x}{\frac{1}{b^2+c^2-a^2}}=\frac{y}{\frac{1}{c^2+a^2-b^2}}=\frac{z}{\frac{1}{a^2+b^2-c^2}}.$$

Put for brevity

$$b^2+c^2-a^2=A,\quad c^2+a^2-b^2=B,\ a^2+b^2-c^2=C.$$

It is evident that our problem is equivalent to the following one: if the equation $x^3+y^3+z^3=(x+y)(x+z)\times$ $\times(y+z)$ has the solution

$$x=a,\quad y=b,\quad z=c,$$

then it also has the following solution

$$x=\frac{1}{A},\quad y=\frac{1}{B},\quad z=\frac{1}{C}.$$

We know the following identity (see Problem 19, Sec. 1).

$$(x+y+z)^3-x^3-y^3-z^3=3(x+y)(x+z)(y+z).$$

Using this identity, we can easily prove that the equalities

$$x^3 + y^3 + z^3 = (x + y)(x + z)(y + z), \quad (1)$$

$$(x + y + z)^3 = 4(x^3 + y^3 + z^3) = $$
$$= 4(x + y)(x + z)(y + z), \quad (2)$$

$$(x + y - z)(x + z - y)(y + z - x) = -4xyz \quad (3)$$

are equivalent, and the existence of any of them involves the existence of the remaining ones. Thus, it is sufficient to prove that

$$\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)^3 = 4\left(\frac{1}{A}+\frac{1}{B}\right)\left(\frac{1}{A}+\frac{1}{C}\right)\left(\frac{1}{B}+\frac{1}{C}\right),$$

i.e. that

$$(AB + AC + BC)^3 = 4(A + B)(A + C)(B + C)\cdot ABC.$$

But

$$A + B = 2c^2, \quad A + C = 2b^2, \quad B + C = 2a^2.$$

Therefore we have to prove

$$(AB + AC + BC)^3 = 32a^2b^2c^2\cdot ABC.$$

Let us first compute $AB + AC + BC$, and then ABC. We have

$$AB + AC + BC = A(B + C) + BC =$$
$$= (b^2 + c^2 - a^2)\cdot 2a^2 + [a^2 + (b^2 - c^2)] \times$$
$$\times [a^2 - (b^2 - c^2)] = 2a^2b^2 + 2a^2c^2 - 2a^4 +$$
$$+ a^4 - b^4 - c^4 + 2b^2c^2 = -a^4 - b^4 - c^4 +$$
$$+ 2a^2b^2 + 2a^2c^2 + 2b^2c^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2 =$$
$$= (a - b + c)(-a + b + c)(a + b - c)(a + b + c).$$

By virtue of equality (3)

$$(a + c - b)(b + c - a)(a + b - c) = -4abc.$$

Therefore

$$AB + AC + BC = -4abc(a + b + c).$$

Compute ABC. Put

$$a^2 + b^2 + c^2 = s,$$

then

$$ABC = (s - 2a^2)(s - 2b^2)(s - 2c^2) =$$
$$= s^3 - 2(a^2 + b^2 + c^2)s^2 + 4(a^2b^2 + a^2c^2 + b^2c^2)s -$$
$$- 8a^2b^2c^2 = 4(a^2b^2 + a^2c^2 + b^2c^2)s - s^3 - 8a^2b^2c^2 =$$
$$= s\{4a^2b^2 + 4a^2c^2 + 4b^2c^2 - (a^2 + b^2 + c^2)^2\} -$$
$$-8a^2b^2c^2 = -s\{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 -$$
$$- 2b^2c^2\} - 8a^2b^2c^2 = s(a + c - b)(b + c - a) \times$$
$$\times (a + b - c)(a + b + c) - 8a^2b^2c^2 =$$
$$= -4abc(a + b + c)(a^2 + b^2 + c^2) - 8a^2b^2c^2 =$$
$$= -4abc\{a^3 + b^3 + c^3 + a^2(b + c) + b^2(a + c) +$$
$$+ c^2(a + b) + 2abc\}.$$

But

$$(a + b)(a + c)(b + c) = a^2(b + c) +$$
$$+ b^2(a + c) + c^2(a + b) + 2abc.$$

Therefore, by virtue of equality (1), the bracketed expression is equal to $2(a^3 + b^3 + c^3)$.

But, by virtue of equality (2),

$$2(a^3 + b^3 + c^3) = \frac{1}{2}(a + b + c)^3.$$

Therefore $ABC = -2abc(a + b + c)^3$.

But, as has been deduced, $AB + AC + BC = -4abc(a + b + c)$.

Therefore,

$$(AB + AC + BC)^3 = 32a^2b^2c^2 \cdot ABC.$$

29. 1° We have:

$$P_n = a_nP_{n-1} + P_{n-2}, \quad P_n - P_{n-2} = a_nP_{n-1},$$
$$Q_n = a_nQ_{n-1} + Q_{n-2}, \quad Q_n - Q_{n-2} = a_nQ_{n-1}.$$

The left member of the equality in question is transformed by the following method

$$\frac{P_{n+2} - P_n}{P_n} \cdot \frac{P_{n+1} - P_{n-1}}{P_{n+1}} = a_{n+2}\frac{P_{n+1}}{P_n} \cdot a_{n+1}\frac{P_n}{P_{n+1}} = a_{n+2} \cdot a_{n+1}.$$

We get quite analogously that the right member also yields $a_{n+1} \cdot a_{n+2}$. Thus, the identity is proved.

2° We have

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = \frac{P_k Q_{k-1} - Q_k P_{k-1}}{Q_k Q_{k-1}} = \frac{(-1)^{k-1}}{Q_k Q_{k-1}}.$$

Putting here $k = 1, 2, \ldots, n$ and adding termwise, we obtain the required result.

3° We have

$$\begin{aligned}
P_{n+2}Q_{n-2} - P_{n-2}Q_{n+2} &= (a_{n+2}P_{n+1} + P_n)\, Q_{n-2} - \\
&- P_{n-2}\,(Q_{n+1}a_{n+2} + Q_n) = a_{n+2}\,(P_{n+1}Q_{n-2} - P_{n-2}Q_{n+1}) + \\
&+ P_nQ_{n-2} - P_{n-2}Q_n = \\
&= a_{n+2}\,\{(a_{n+1}P_n + P_{n-1})\, Q_{n-2} - P_{n-2}\,(a_{n+1}Q_n + Q_{n-1})\} + \\
&+ (a_nP_{n-1} + P_{n-2})\, Q_{n-2} - P_{n-2}\,(a_nQ_{n-1} + Q_{n-2}) = \\
&= a_{n+1}a_{n+2}\,(P_nQ_{n-2} - P_{n-2}Q_n) + \\
&+ a_{n+2}\,(P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1}) + \\
&+ a_n\,(P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1}) = \\
&= a_{n+1}a_{n+2}\,\{(a_nP_{n-1} + P_{n-2})\, Q_{n-2} - \\
&- P_{n-2}\,(a_nQ_{n-1} + Q_{n-2})\} + a_{n+2}(-1)^n + a_n\,(-1)^n = \\
&= (a_{n+2}a_{n+1}a_n + a_{n+2} + a_n)\,(-1)^n.
\end{aligned}$$

4° It is known that $P_n = a_nP_{n-1} + P_{n-2}$. Therefore

$$\frac{P_n}{P_{n-1}} = a_n + \frac{P_{n-2}}{P_{n-1}} = a_n + \frac{1}{\frac{P_{n-1}}{P_{n-2}}} = a_n + \frac{1}{\frac{a_{n-1}P_{n-2} + P_{n-3}}{P_{n-2}}} =$$

$$= a_n + \frac{1}{a_{n-1} + \frac{P_{n-3}}{P_{n-2}}} = a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_2 + \frac{P_0}{P_1}} =$$

$$= a_n + \frac{1}{a_{n-1}} + \cdots + \frac{1}{a_1 + \frac{1}{a_0}}.$$

The expression for $\frac{Q_n}{Q_{n-1}}$ is found in a similar way.

30. On the basis of the results of the preceding problem we have

$$\frac{P_n}{P_{n-1}}=(a_n,\ a_{n-1},\ \ldots,\ a_0)=(a_0,\ a_2,\ \ldots,\ a_n)=\frac{P_n}{Q_n}.$$

Consequently, $P_{n-1} = Q_n$.

31. We have to prove that

$$P_{n+1}^2-P_{n-1}P_{n+1}=P_nP_{n+2}-P_n^2,$$

or

$$P_{n+1}(P_{n+1}-P_{n-1})=P_n(P_{n+2}-P_n).$$

But

$$P_{n+1}=aP_n+P_{n-1},\quad P_{n+2}=aP_{n+1}+P_n.$$

Consequently,

$$P_{n+1}-P_{n-1}=aP_n,\quad P_{n+2}-P_n=aP_{n+1}.$$

Hence, follows the validity of our identity.

32. By hypothesis

$$x=\frac{1}{(a,\ b,\ \ldots,\ l,\ a,\ b,\ \ldots,\ l)}.\quad \frac{P_n}{Q_n}=\frac{1}{(a,\ b,\ \ldots,\ l)}$$

Or

$$x=\cfrac{1}{a+\cfrac{1}{b+\ddots+\cfrac{1}{l+\cfrac{P_n}{Q_n}}}}.$$

Thus, x is obtained from $\frac{P_n}{Q_n}$ if l is replaced by $l+\frac{P_n}{Q_n}$ in this fraction. But $\frac{P_n}{Q_n}=\frac{lP_{n-1}+P_{n-2}}{lQ_{n-1}+Q_{n-2}}$. Therefore

$$x=\frac{\left(l+\frac{P_n}{Q_n}\right)P_{n-1}+P_{n-2}}{\left(l+\frac{P_n}{Q_n}\right)Q_{n-1}+Q_{n-2}}=\frac{P_nQ_n+P_nP_{n-1}}{Q_n^2+P_nQ_{n-1}}.$$

33. It is obvious that at $k=0,\ 1$ our formula holds true. Assuming that it is valid at $k=n-1$, let us prove that it takes place also at $k=n$. And so, we assume

$$b_0+\cfrac{a_1}{b_1+\ddots+\cfrac{a_{n-1}}{b_{n-1}}}=\frac{P_{n-1}}{Q_{n-1}}.$$

However, according to the rule for composing P_k and Q_k, we have

$$\frac{P_{n-1}}{Q_{n-1}}=\frac{b_{n-1}P_{n-2}+a_{n-1}P_{n-3}}{b_{n-1}Q_{n-2}+a_{n-1}Q_{n-3}},$$

where P_{n-2}, P_{n-3}, Q_{n-2}, Q_{n-3} are independent of a_{n-1} and b_{n-1}.

On the other hand, it is clear that the fraction

$$b_0+\frac{a_1}{b_1+\ddots+\frac{a_{n-1}}{b_{n-1}+\frac{a_n}{b_n}}}$$

is obtained from the fraction

$$b_0+\frac{a_1}{b_1+\ddots+\frac{a_{n-1}}{b_{n-1}}}$$

by replacing b_{n-1} by $b_{n-1}+\frac{a_n}{b_n}$.

Therefore

$$b_0+\frac{a_1}{b_1+\ddots+\frac{a_{n-1}}{b_{n-1}+\frac{a_n}{b_n}}}=\frac{\left(b_{n-1}+\frac{a_n}{b_n}\right)P_{n-2}+a_{n-1}P_{n-3}}{\left(b_{n-1}+\frac{a_n}{b_n}\right)Q_{n-2}+a_{n-1}Q_{n-3}}=$$

$$=\frac{b_{n-1}P_{n-2}+a_{n-1}P_{n-3}+\frac{a_n}{b_n}P_{n-2}}{b_{n-1}Q_{n-2}+a_{n-1}Q_{n-3}+\frac{a_n}{b_n}Q_{n-2}}=\frac{P_{n-1}+\frac{a_n}{b_n}P_{n-2}}{Q_{n-1}+\frac{a_n}{b_n}Q_{n-2}}=$$

$$=\frac{b_nP_{n-1}+a_nP_{n-2}}{b_nQ_{n-1}+a_nQ_{n-2}}=\frac{P_n}{Q_n}.$$

34. Denoting the value of our fraction by $\frac{P_n}{Q_n}$, we have

$$P_1=r, \qquad Q_1=r+1,$$
$$P_2=r(r+1), \quad Q_2=r^2+r+1.$$

Using the method of induction, let us prove that

$$P_n=r\frac{r^n-1}{r-1}, \quad Q_n=\frac{r^{n+1}-1}{r-1}.$$

At $n = 1$ these formulas are valid. Assuming their validity at $n = m$, let us prove that they also take place at $n = m + 1$.

We have

$$P_{m+1} = b_{m+1}P_m + a_{m+1}P_{m-1}.$$

In our case we find

$$P_{m+1} = (r+1)\, r\frac{r^m - 1}{r-1} - r^2\frac{r^{m-1}-1}{r-1} = r\frac{r^{m+1}-1}{r-1}.$$

Analogously we obtain that

$$Q_{m+1} = \frac{r^{m+2}-1}{r-1}.$$

35. Put

$$\frac{1}{u_r} + \frac{1}{u_{r+1}} = \frac{1}{u_r + x_r}.$$

Then we find

$$x_r = -\frac{u_r^2}{u_r + u_{r+1}}.$$

Therefore

$$\frac{1}{u_1} + \frac{1}{u_2} = \cfrac{1}{u_1 - \cfrac{u_1^2}{u_1 + u_2}}.$$

Further

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} = \frac{1}{u_1} + \frac{1}{u_2 + x_2} = \frac{1}{u_1 + x_2'},$$

where

$$x_2' = -\frac{u_1^2}{u_1 + u_2 + x_2}.$$

Thus

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} = \cfrac{1}{u_1 - \cfrac{u_1^2}{u_1 + u_2 + x_2}} = \cfrac{1}{u_1 - \cfrac{u_1^2}{u_1 + u_2 - \cfrac{u_2^2}{u_2 + u_3}}}.$$

Using the method of induction, we also get the general formula.

36. Let us denote the fraction

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cdots+\cfrac{a_n}{b_n}}}$$

by $\frac{P_n}{Q_n}$, and put the fraction

$$\cfrac{c_1a_1}{c_1b_1+\cfrac{c_1c_2a_2}{c_2b_2+\cdots+\cfrac{c_{n-1}c_na_n}{c_nb_n}}}$$

equal to $\frac{P'_n}{Q'_n}$. It is required to prove that $\frac{P_n}{Q_n}=\frac{P'_n}{Q'_n}$ for any whole positive n.

We have

$$\frac{P_1}{Q_1}=\frac{a_1}{b_1},\quad \frac{P_2}{Q_2}=\frac{a_1b_2}{b_1b_2+a_2},\ \ldots;$$
$$\frac{P'_1}{Q'_1}=\frac{c_1a_1}{c_1b_1},\quad \frac{P'_2}{Q'_2}=\frac{c_1c_2a_1b_2}{c_1c_2(b_1b_2+a_2)},\ \ldots .$$

We may put $P_1=a_1$, $Q_1=b_1$, $P_2=a_1b_2$, $Q_2=b_1b_2+a_2$, and then the following relations take place (see Problem 33)

$$P_{n+1}=b_{n+1}P_n+a_{n+1}P_{n-1},$$
$$Q_{n+1}=b_{n+1}Q_n+a_{n+1}Q_{n-1}.$$

Put

$$P'_1=c_1a_1,\quad P'_2=c_1c_2a_1b_2;$$
$$Q'_1=c_1b_1,\quad Q'_2=c_1c_2(b_1b_2+a_2)$$

Let us prove that for any n we then have

$$P'_n=c_1c_2\ldots c_nP_n,\quad Q'_n=c_1c_2\ldots c_nQ_n.$$

Let us prove this assertion using the method of induction, i.e. assuming its validity for a subscript smaller than, or equal to, n, we shall prove the validity for the subscript $n+1$.

We have

$$P'_{n+1}=c_{n+1}b_{n+1}P'_n+c_nc_{n+1}a_{n+1}P'_{n-1},$$
$$Q'_{n+1}=c_{n+1}b_{n+1}Q'_n+c_nc_{n+1}a_{n+1}Q'_{n-1}.$$

Hence (with the asumption)

$$P'_{n+1} = c_{n+1}b_{n+1}c_1c_2 \dots c_nP_n + c_nc_{n+1}a_{n+1}c_1c_2 \dots c_{n-1}P_{n-1} = c_1c_2 \dots c_{n+1}(b_{n+1}P_n + a_{n+1}P_{n-1}) = c_1c_2 \dots c_{n+1}P_{n+1}$$

Likewise prove that

$$Q'_{n+1} = c_1c_2 \dots c_{n+1}Q_{n+1}.$$

Now it is easy to find that

$$\frac{P_n}{Q_n} = \frac{P'_n}{Q'_n}.$$

37. 1° Put

$$2\cos x - \cfrac{1}{2\cos x - \cfrac{1}{2\cos x - \dots - \cfrac{1}{2\cos x}}} = \frac{P_n}{Q_n}.$$

We have

$$\frac{P_1}{Q_1} = 2\cos x.$$

Therefore we may put

$$P_1 = \frac{\sin 2x}{\sin x}, \quad Q_1 = \frac{\sin x}{\sin x}.$$

Further

$$\frac{P_2}{Q_2} = 2\cos x - \frac{1}{2\cos x} = \frac{4\cos^2 x - 1}{2\cos x}.$$

Consequently, we may take

$$P_2 = \frac{\sin 3x}{\sin x}, \quad Q_2 = \frac{\sin 2x}{\sin x}.$$

Let us prove that then $P_n = \frac{\sin (n+1)x}{\sin x}$, $Q_n = \frac{\sin nx}{\sin x}$ for any n.

Assuming that these formulas are valid for subscripts not exceeding n, let us prove that they also take place at $n+1$. We have (see Problem 33)

$$P_{n+1} = 2\cos x \frac{\sin (n+1)x}{\sin x} - \frac{\sin nx}{\sin x} = \frac{1}{\sin x}\sin (n+2)x.$$

In the same way we find that $Q_{n+1}=\frac{\sin(n+1)x}{\sin x}$, and therefore

$$\frac{P_n}{Q_n}=\frac{\sin(n+1)x}{\sin nx}$$

for any whole positive n.

2° Let us denote the continued fraction on the right by $\frac{P_n}{Q_n}$. We have to prove that

$$\frac{P_n}{Q_n}=1+b_2+b_2b_3+\ldots+b_2b_3\ldots b_n.$$

We have

$$\frac{P_1}{Q_1}=\frac{1}{1}, \quad \frac{P_2}{Q_2}=\frac{b_2+1}{1}.$$

Therefore we may take: $P_1=1$, $Q_1=1$, $P_2=b_2+1$, $Q_2=1$. Then, using the method of induction, it is easy to prove that

$$P_n=1+b_2+b_2b_3+\ldots+b_2b_3\ldots b_n,$$
$$Q_n=1,$$

and, consequently, our equality is also true.

38. 1° We have

$$\sin a+\sin b+\sin c=\sin(a+b+c)=$$
$$=(\sin a+\sin b)+[\sin c-\sin(a+b+c)]=$$
$$=2\sin\frac{a+b}{2}\cos\frac{a-b}{2}-2\sin\frac{a+b}{2}\cos\frac{a+b+2c}{2}=$$
$$=2\sin\frac{a+b}{2}\left(\cos\frac{a-b}{2}-\cos\frac{a+b+2c}{2}\right)=$$
$$=4\sin\frac{a+b}{2}\sin\frac{a+c}{2}\sin\frac{b+c}{2}\Big).$$

2° Analogous to the preceding one.

39. Consider the sum

$$\tan a+\tan b+\tan c.$$

We have

$$\tan a+\tan b+\tan c=\frac{\sin(a+b)}{\cos a\cos b}+\frac{\sin c}{\cos c}=$$

$$=\frac{\sin(a+b)\cos c+\sin c\cos a\cos b}{\cos a\cos b\cos c}=$$

$$=\frac{\sin(a+b)\cos c+\cos(a+b)\sin c-\cos(a+b)\sin c+\sin c\cos a\cos b}{\cos a\cos b\cos c}=$$

$$=\frac{\sin(a+b+c)+\sin c\,[\cos a\cos b-\cos(a+b)]}{\cos a\cos b\cos c}=$$

$$=\frac{\sin(a+b+c)+\sin a\sin b\sin c}{\cos a\cos b\cos c}.$$

Hence follows the required equality.

40. The equalities 1°, 2° and 3° are easily obtained from Problems 38 (1°, 2°) and 39 putting $a=A$, $b=B$, $c=C$ and $a+b+c=A+B+C=\pi$.

Now let us prove 4°. Rewrite the left member in the following way

$$S=\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right).$$

But since

$$A+B+C=\pi,$$

we have

$$\tan\frac{C}{2}=\tan\left(\frac{\pi}{2}-\frac{A+B}{2}\right)=\cot\frac{A+B}{2}=\frac{1}{\tan\frac{A+B}{2}}.$$

Hence

$$S=\tan\frac{A}{2}\tan\frac{B}{2}+\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A+B}{2}}=1,$$

since

$$\tan\frac{A+B}{2}=\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}.$$

5° Indeed

$$\sin 2A + \sin 2B + \sin 2C =$$
$$= \sin 2A + 2 \sin (B + C) \cos (B - C) =$$
$$= 2 \sin A \cos A + 2 \sin A \cos (B - C) =$$
$$= 2 \sin A [\cos A + \cos (B - C)] =$$
$$= 4 \sin A \sin B \sin C.$$

41. 1° It is necessary to find how a, b, and c are related if

$$\cos a + \cos b + \cos c - 1 - 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} = 0.$$

To this end let us reduce the left member of the equality to a form convenient for taking logs, i.e. try to represent it in the form of a product of trigonometric functions of the quantities a, b and c.

We have

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2} =$$
$$= 2 \left(\cos^2 \frac{a}{2} \cos^2 \frac{b}{2} - \sin^2 \frac{a}{2} \sin^2 \frac{b}{2}\right),$$
$$\cos c - 1 = -2 \sin^2 \frac{c}{2}.$$

Therefore the left member takes the form

$$2 \cos^2 \frac{a}{2} \cos^2 \frac{b}{2} - 2 \sin^2 \frac{a}{2} \sin^2 \frac{b}{2} - 2 \sin^2 \frac{c}{2} -$$
$$- 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} =$$
$$= 2 \left[\cos^2 \frac{a}{2} \cos^2 \frac{b}{2} - \left(\sin^2 \frac{a}{2} \sin^2 \frac{b}{2} + 2 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} + \right.\right.$$
$$\left.\left. + \sin^2 \frac{c}{2}\right)\right] = 2 \left[\cos^2 \frac{a}{2} \cos^2 \frac{b}{2} - \left(\sin \frac{a}{2} \sin \frac{b}{2} + \sin \frac{c}{2}\right)^2\right] =$$
$$= 2 \left[\left(\cos \frac{a}{2} \cos \frac{b}{2} + \sin \frac{a}{2} \sin \frac{b}{2}\right) + \sin \frac{c}{2}\right] \times$$
$$\times \left[\left(\cos \frac{a}{2} \cos \frac{b}{2} - \sin \frac{a}{2} \sin \frac{b}{2}\right) - \sin \frac{c}{2}\right] =$$
$$= 2 \left(\cos \frac{a-b}{2} + \sin \frac{c}{2}\right) \left(\cos \frac{a+b}{2} - \sin \frac{c}{2}\right) =$$

$$= 2\left[\cos\frac{a-b}{2} + \cos\left(\frac{\pi}{2} - \frac{c}{2}\right)\right]\left[\cos\frac{a+b}{2} - \cos\left(\frac{\pi}{2} - \frac{c}{2}\right)\right] =$$

$$= -8\sin\frac{\pi+b+c-a}{4}\sin\frac{\pi+a+c-b}{4}\times$$

$$\times\sin\frac{\pi+a+b-c}{4}\sin\frac{a+b+c-\pi}{4}.$$

By hypothesis, this expression must equal zero and, consequently, at least one of the factors must be equal to zero. But from the equality $\sin\alpha = 0$ follows $\alpha = k\pi$ (where k is any whole number). Therefore, among a, b and c, satisfying the original relationship, there exists at least one of the four relationships

$$a + b + c = (4k + 1)\pi, \quad a + b - c = (4k - 1)\pi,$$

$$a + c - b = (4k - 1)\pi, \quad b + c - a = (4k - 1)\pi.$$

2° We have (see Problem 30)

$$\tan a + \tan b + \tan c - \tan a \tan b \tan c = \frac{\sin(a+b+c)}{\cos a \cos b \cos c}.$$

By virtue of our conditions

$$\sin(a + b + c) = 0 \text{ and } a + b + c = k\pi.$$

3° Transform the original expression. We have

$$1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c =$$
$$= 1 - \cos^2 a - \cos^2 b - (\cos^2 c - 2\cos a\cos b\cos c +$$
$$+ \cos^2 a\cos^2 b) + \cos^2 a\cos^2 b = 1 - \cos^2 a - \cos^2 b -$$
$$- (\cos c - \cos a\cos b)^2 + \cos^2 a\cos^2 b =$$
$$= (1 - \cos^2 a)(1 - \cos^2 b) - (\cos c - \cos a\cos b)^2 =$$
$$= (\sin a\sin b - \cos c + \cos a\cos b)\times$$
$$\times(\sin a\sin b + \cos c - \cos a\cos b) =$$
$$= [\cos c - \cos(a + b)][\cos(a - b) - \cos c] =$$
$$= 4\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}\sin\frac{c+b-a}{2}.$$

Consequently, there exists at least one of the following relations

$$a + b + c = 2k\pi, \quad a + b - c = 2k\pi, \quad a + c - b = 2k\pi,$$
$$b + c - a = 2k\pi.$$

42. Put

$$x = \tan\frac{\alpha}{2}, \quad y = \tan\frac{\beta}{2}, \quad z = \tan\frac{\gamma}{2}.$$

Then

$$\frac{2x}{1-x^2}=\tan\alpha,\quad \frac{2y}{1-y^2}=\tan\beta,\quad \frac{2z}{1-z^2}=\tan\gamma,$$

and our problem takes the following form. Prove that

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$$

if

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=1.$$

Rewrite the last equality as

$$\tan\frac{\alpha}{2}\left(\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right)-\left(1-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)=0.$$

Dividing both members by $1-\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$, we get

$$\tan\frac{\alpha}{2}\tan\frac{\beta+\gamma}{2}-1=0,\quad \tan\frac{\alpha}{2}=\cot\frac{\beta+\gamma}{2}=\tan\left(\frac{\pi}{2}-\frac{\beta+\gamma}{2}\right).$$

Hence

$$\frac{\alpha}{2}+\frac{\beta+\gamma}{2}-\frac{\pi}{2}=k\pi$$

(if tangents are equal, the corresponding angles differ by the multiple of π) and

$$\alpha+\beta+\gamma=(2k+1)\pi.$$

And so the proposition is proved (see Problem 40, 3°).

43. Put $b=\tan\beta$, $c=\tan\gamma$, $a=\tan\alpha$. Then

$$\frac{b-c}{1+bc}=\frac{\tan\beta-\tan\gamma}{1+\tan\beta\tan\gamma}=\tan(\beta-\gamma),$$

and, hence, our equality is equivalent to the following one

$$\tan(\beta-\gamma)+\tan(\gamma-\alpha)+\tan(\alpha-\beta)=$$
$$=\tan(\beta-\gamma)\tan(\gamma-\alpha)\tan(\alpha-\beta).$$

Put

$$\beta-\gamma=x,\quad \gamma-\alpha=y,\quad \alpha-\beta=z.$$

Let us finally prove that

$$\tan x+\tan y+\tan z=\tan x\tan y\tan z$$

if

$$x + y + z = 0.$$

But then we have

$$\tan(x+y) = -\tan z, \qquad \frac{\tan x + \tan y}{1 - \tan x \tan y} = -\tan z.$$

Hence follows the required equality.

It is obvious, that the last two problems can be solved by direct transformations of the considered algebraic expressions.

44. We have

$$\tan 3\alpha = \frac{\sin 3\alpha}{\cos 3\alpha} = \frac{\sin\alpha\,(3-4\sin^2\alpha)}{\cos\alpha\,(1-4\sin^2\alpha)} = \tan\alpha\,\frac{3-4\sin^2\alpha}{1-4\sin^2\alpha}.$$

Divide both the numerator and denominator of this fraction by $\cos^2\alpha$ and replace $\frac{1}{\cos^2\alpha}$ by $1+\tan^2\alpha$.

We get

$$\tan 3\alpha = \tan\alpha\,\frac{3-\tan^2\alpha}{1-3\tan^2\alpha} = \tan\alpha\,\frac{\sqrt{3}+\tan\alpha}{1-\sqrt{3}\tan\alpha}\cdot\frac{\sqrt{3}-\tan\alpha}{1+\sqrt{3}\tan\alpha}.$$

Hence

$$\tan 3\alpha = \tan\alpha\,\tan\left(\frac{\pi}{3}+\alpha\right)\tan\left(\frac{\pi}{3}- \quad\right).$$

45. Multiplying both members of the equality by $a+b$ and replacing unity in the right member by $(\sin^2\alpha + \cos^2\alpha)^2$, we get

$$\sin^4\alpha + \cos^4\alpha + \frac{b}{a}\sin^4\alpha + \frac{a}{b}\cos^4\alpha = \sin^4\alpha + \cos^4\alpha + 2\sin^2\alpha\cos^2\alpha,$$

whence

$$\frac{b}{a}\sin^4\alpha - 2\sin^2\alpha\cos^2\alpha + \frac{a}{b}\cos^4\alpha = 0,$$

$$\left(\sqrt{\frac{b}{a}}\sin^2\alpha - \sqrt{\frac{a}{b}}\cos^2\alpha\right)^2 = 0,$$

$$\frac{b}{a}\sin^4\alpha = \frac{a}{b}\cos^4\alpha,$$

or

$$\frac{\sin^4\alpha}{a^2}=\frac{\cos^4\alpha}{b^2}=\lambda.$$

Substituting it into the original equality, we find

$$\lambda=\frac{1}{(a+b)^2}.$$

Therefore

$$\frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}=\frac{1}{(a+b)^3}.$$

46. From the second equality we have

$$(a_1\cos\alpha_1+a_2\cos\alpha_2+\ldots+a_n\cos\alpha_n)\cos\theta-$$
$$-(a_1\sin\alpha_1+a_2\sin\alpha_2+\ldots+a_n\sin\alpha_n)\sin\theta=0.$$

On the basis of the first equality and since $\sin\theta\neq 0$, we get

$$a_1\sin\alpha_1+a_2\sin\alpha_2+\ldots+a_n\sin\alpha_n=0. \qquad (*)$$

Multiplying the first equality by $\cos\lambda$ and the equality (*) by $\sin\lambda$, and subtracting the second result from the first one, we have

$$a_1\cos(\alpha_1+\lambda)+a_2\cos(\alpha_2+\lambda)+\ldots+$$
$$+a_n\cos(\alpha_n+\lambda)=0.$$

47. It is obvious that the left member is reduced to the following expression

$$(\tan\beta-\tan\gamma)+(\tan\gamma-\tan\alpha)+(\tan\alpha-\tan\beta)=0.$$

48. 1° We have

$$r_a-r=\frac{s}{p-a}-\frac{s}{p}=\frac{sa}{p(p-a)}.$$

Hence

$$\frac{a^2}{r_a-r}=\frac{ap(p-a)}{s}.$$

Therefore

$$\omega=\frac{a^2}{r_a-r}+\frac{b^2}{r_b-r}+\frac{c^2}{r_c-r}=\frac{p}{s}\{a(p-a)+b(p-b)+c(p-c)\}.$$

But

$$s^2 = p(p-a)(p-b)(p-c).$$

Hence

$$\omega = s\left\{\frac{a}{(p-b)(p-c)}+\frac{b}{(p-a)(p-c)}+\frac{c}{(p-a)(p-b)}\right\} =$$

$$= s\left\{\frac{(p-b)+(p-c)}{(p-b)(p-c)}+\frac{(p-a)+(p-c)}{(p-a)(p-c)}+\frac{(p-a)+(p-b)}{(p-a)(p-b)}\right\} =$$

$$= 2(r_a+r_b+r_c).$$

2° We have

$$\sigma = \frac{a^2r_a}{(a-b)(a-c)}+\frac{b^2r_b}{(b-c)(b-a)}+\frac{c^2r_c}{(c-a)(c-b)} =$$

$$= s\left\{\frac{a^2}{(p-a)(a-b)(a-c)}+\frac{b^2}{(p-b)(b-c)(b-a)}+\right.$$

$$\left.+\frac{c^2}{(p-c)(c-a)(c-b)}\right\}.$$

But (see Problem 9)

$$\frac{a^2}{(p-a)(a-b)(a-c)}+\frac{b^2}{(p-b)(b-c)(b-a)}+$$

$$+\frac{c^2}{(p-c)(c-a)(c-b)} = \frac{p^2}{(p-a)(p-b)(p-c)}.$$

Therefore

$$\sigma = \frac{sp^2}{(p-a)(p-b)(p-c)} = \frac{sp^3}{s^2} = \frac{p^3}{s} = \frac{p^2}{r}.$$

3° We get

$$r_a+r_b+r_c = s\left(\frac{1}{p-a}+\frac{1}{p-b}+\frac{1}{p-c}\right) = \frac{s(ab+ac+bc-p^2)}{(p-a)(p-b)(p-c)}.$$

Further

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c} = \frac{1}{s}\{a(p-a)+b(p-b)+c(p-c)\} =$$

$$= \frac{1}{s}(2p^2-a^2-b^2-c^2) =$$

$$= \frac{2}{s}(-p^2+ab+ac+bc).$$

The rest is obvious.

4° Consider the first sum

$$\sigma = \frac{1}{s^2}\left\{\frac{bc\,(p-a)^2}{(a-b)\,(a-c)} + \frac{ac\,(p-b)^2}{(b-c)\,(b-a)} + \frac{ab\,(p-c)^2}{(c-a)\,(c-b)}\right\} =$$

$$= \frac{1}{s^2}\left\{p^2\left[\frac{bc}{(a-b)\,(a-c)} + \frac{ac}{(b-c)\,(b-a)} + \frac{ab}{(c-a)\,(c-b)}\right] -\right.$$

$$-2pabc\left[\frac{1}{(a-b)\,(a-c)} + \frac{1}{(b-c)\,(b-a)} + \frac{1}{(c-a)\,(c-b)}\right] +$$

$$\left. + abc\left[\frac{a}{(a-b)\,(a-c)} + \frac{b}{(b-c)\,(b-a)} + \frac{c}{(c-a)\,(c-b)}\right]\right\}.$$

But (see Problem 8)

$$\frac{1}{(a-b)\,(a-c)} + \frac{1}{(b-c)\,(b-a)} + \frac{1}{(c-a)\,(c-b)} = 0,$$

$$\frac{a}{(a-b)\,(a-c)} + \frac{b}{(b-c)\,(b-a)} + \frac{c}{(c-a)\,(c-b)} = 0.$$

Therefore

$$\sigma = \frac{p^2}{s^2}\left[\frac{bc}{(a-b)\,(a-c)} + \frac{ac}{(b-c)\,(b-a)} + \frac{ab}{(c-a)\,)c-b)}\right];$$

further

$$\frac{bc}{(a-b)\,(a-c)} + \frac{ac}{(b-c)\,(b-a)} + \frac{ab}{(c-a)\,(c-b)} =$$

$$= abc\left\{\left[\frac{1}{a\,(a-b)\,(a-c)} + \frac{1}{b\,(b-c)\,(b-a)} + \frac{1}{c\,(c-a)\,(c-b)} +\right.\right.$$

$$\left.\left. + \frac{1}{(0-a)\,(0-b)\,(0-c)}\right] + \frac{1}{abc}\right\} = 1.$$

And so

$$\sigma = \frac{p^2}{s^2} = \frac{1}{r^2}\,.$$

Let us go over to the second sum. We have

$$\sigma = \frac{1}{r_a r_b r_c}\left\{\frac{a^2 r_a}{(a-b)\,(a-c)} + \frac{b^2 r_b}{(b-c)\,(b-a)} +\right.$$

$$\left. + \frac{c^2 r_c}{(c-a)\,(c-b)}\right\} = \frac{s}{r_a r_b r_c}\left\{\frac{a^2}{(a-b)\,(a-c)\,(p-a)} +\right.$$

$$\left. + \frac{b^2}{(b-c)\,(b-a)\,(p-b)} + \frac{c^2}{(c-a)\,(c-b)\,(p-c)}\right\}.$$

But

$$\frac{a^2}{(a-b)(a-c)(a-p)}+\frac{b^2}{(b-c)(b-a)(b-p)}+$$
$$+\frac{c^2}{(c-a)(c-b)(c-p)}+\frac{p^2}{(p-a)(p-b)(p-c)}=0.$$

Therefore

$$\sigma=\frac{s(p-a)(p-b)(p-c)}{s^3}\cdot\frac{p^2}{(p-a)(p-b)(p-c)}=\frac{p^2}{s^2}=\frac{1}{r^2}.$$

5° We have

$$\sigma=\frac{ar_a}{(a-b)(a-c)}+\frac{br_b}{(b-c)(b-a)}+\frac{cr_c}{(c-a)(c-b)}=$$
$$=s\left\{\frac{a}{(a-b)(a-c)(p-a)}+\frac{b}{(b-c)(b-a)(p-b)}+\right.$$
$$\left.+\frac{c}{(c-a)(c-b)(p-c)}\right\}=-s\left\{\frac{a}{(a-b)(a-c)(a-p)}+\right.$$
$$+\frac{b}{(b-c)(b-a)(b-p)}+\frac{c}{(c-a)(c-b)(c-p)}+$$
$$\left.+\frac{p}{(p-a)(p-b)(p-c)}-\frac{p}{(p-a)(p-b)(p-c)}\right\}=$$
$$=\frac{sp}{(p-a)(p-b)(p-c)}=\frac{p^2}{s}=\frac{p}{r}.$$

Further

$$\sigma=\frac{(b+c)r_a}{(a-b)(a-c)}+\frac{(c+a)r_b}{(b-c)(b-a)}+\frac{(a+b)r_c}{(c-a)(c-b)}=$$
$$=s\left\{\frac{(b+c)}{(a-b)(a-c)(p-a)}+\frac{(c+a)}{(b-c)(b-a)(p-b)}+\right.$$
$$\left.+\frac{(a+b)}{(c-a)(c-b)(p-c)}\right\}=s(a+b+c)\left\{\frac{1}{(a-b)(a-c)(p-a)}+\right.$$
$$\left.+\frac{1}{(b-c)(b-a)(p-b)}+\frac{1}{(c-a)(c-b)(p-c)}\right\}-$$
$$-s\left\{\frac{a}{(a-b)(a-c)(p-a)}+\frac{b}{(b-c)(b-a)(p-b)}+\right.$$
$$\left.+\frac{c}{(c-a)(c-b)(p-c)}\right\}.$$

But

$$\frac{1}{(a-b)(a-c)(a-p)}+\frac{1}{(b-c)(b-a)(b-p)}+$$
$$+\frac{1}{(c-a)(c-b)(c-p)}+\frac{1}{(p-a)(p-b)(p-c)}=0.$$

Therefore, the first braced expression is equal to $\frac{1}{(p-a)(p-b)(p-c)}$. The second braced expression is equal to $\frac{p^2}{s^2}$. Hence

$$\sigma=\frac{s(a+b+c)}{(p-a)(p-b)(p-c)}-\frac{p^2}{s}=\frac{2p^2}{s}-\frac{p^2}{s}=\frac{p^2}{s}=\frac{p}{r}.$$

49. Rewrite the supposed identity in the following way:

$$\sin(a+b-c-d)\sin(a-b)=$$
$$=\sin(a-c)\sin(a-d)-\sin(b-c)\sin(b-d).$$

Using the formula $\sin A \sin B=\frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$, we find

$$\sin(a+b-c-d)\sin(a-b)=$$
$$=\frac{1}{2}\{\cos(2b-c-d)-\cos(2a-c-d)\},$$

$$\sin(a-c)\sin(a-d)=\frac{1}{2}\{\cos(c-d)-\cos(2a-c-d)\},$$

$$\sin(b-c)\sin(b-d)=\frac{1}{2}\{\cos(c-d)-\cos(2b-c-d)\}.$$

The rest is obvious.

50. 1° We have: $1+\tan^2\frac{\theta}{2}=\frac{1}{\cos^2\frac{\theta}{2}}=\frac{2}{1+\cos\theta}=\frac{b+c}{p}$,

where $a+b+c=2p$.

Hence

$$1+\tan^2\frac{\theta}{2}+1+\tan^2\frac{\varphi}{2}+1+\tan^2\frac{\psi}{2}=$$
$$=\frac{(b+c)+(a+c)+(a+b)}{p}=4,$$

and, consequently, $\tan^2\frac{\theta}{2}+\tan^2\frac{\varphi}{2}+\tan^2\frac{\psi}{2}=1$.

2° $\tan^2\frac{\theta}{2}=\frac{b+c}{p}-1=\frac{p-a}{p}$. Therefore

$$\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\tan\frac{\psi}{2}=\sqrt{\frac{(p-a)(p-b)(p-c)}{p^3}}.$$

But, as is known

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\sqrt{\frac{(p-a)(p-b)(p-c)}{p^3}}.$$

Hence, $\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\tan\frac{\psi}{2}=\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

51. The left member of our equality can be rewritten as

$$\frac{1}{\sin(a-b)\sin(a-c)\sin(b-c)}\{\sin(b-c)-\sin(a-c)+\sin(a-b)\}.$$

But we have

$$\sin(b-c)-\sin(a-c)=2\sin\frac{b-a}{2}\cos\frac{b+a-2c}{2}.$$

Therefore, the braced expression is equal to

$$2\sin\frac{b-a}{2}\cos\frac{b+a-2c}{2}-2\sin\frac{b-a}{2}\cos\frac{b-a}{2}=$$
$$=4\sin\frac{b-a}{2}\sin\frac{b-c}{2}\sin\frac{c-a}{2}.$$

But

$$\sin(a-b)\sin(a-c)\sin(b-c)=$$
$$=8\sin\frac{a-b}{2}\sin\frac{a-c}{2}\sin\frac{b-c}{2}\cos\frac{a-b}{2}\cos\frac{a-c}{2}\cos\frac{b-c}{2}.$$

The rest is obvious.

52. 1° The fraction in the left member has the form

$$\frac{1}{\sin(a-b)\sin(a-c)\sin(b-c)}\{\sin a\sin(b-c)+\sin b\sin(c-a)+\sin c\sin(a-b)\}=$$
$$=\frac{1}{\sin(a-b)\sin(a-c)\sin(b-c)}\cdot\sum\sin a\sin(b-c),$$

where summing is applied to all the expressions obtained from the one under the summation sign by means of a circular permutation. But

$$\sin a \sin (b-c)=\frac{1}{2}[\cos (a-b+c)-\cos (a+b-c)].$$

Therefore we have

$$\sum \sin a \sin (b-c)=\frac{1}{2}\{\cos (a+c-b)-\cos (a+b-c)+$$
$$+\cos (b+a-c)-\cos (b+c-a)+\cos (c+b-a)-$$
$$-\cos (c+a-b)\}=0,$$

and our identity holds true.

2° The given identity can be proved similarly to case 1°. But we can get the same formula immediately from formula 1°, replacing a by $\frac{\pi}{2}-a$, b by $\frac{\pi}{2}-b$, and, finally, c by $\frac{\pi}{2}-c$.

53. 1° We have to prove that $\sum \sin a \sin (b-c) \times \cos (b+c-a)=0$. Here summation is applied to all the expressions obtained from the original one by means of a circular permutation. But

$$\sin a \sin (b-c)=\frac{1}{2}\{\cos (a-b+c)-\cos (a+b-c)\}.$$

Therefore

$$\sum \sin a \sin (b-c) \cos (b+c-a)=\frac{1}{2} \sum \cos (b+c-a) \times$$
$$\times \cos (a-b+c)-\frac{1}{2} \sum \cos (a+b-c) \cos (b+c-a)=$$
$$=\frac{1}{4} \sum[\cos 2c+\cos (2b-2a)-\cos 2b-\cos (2c-2a)]=$$
$$=\frac{1}{4}\{\cos 2c-\cos 2b+\cos 2a-\cos 2c+\cos 2b-$$
$$-\cos 2a+\cos (2b-2a)-\cos (2c-2a)+\cos (2c-2b)-$$
$$-\cos (2a-2b)+\cos (2a-2c)-\cos (2b-2c)\}=0.$$

2° Can be obtained from 1° by replacing a by $\frac{\pi}{2}-a$, b by $\frac{\pi}{2}-b$ and c by $\frac{\pi}{2}-c$.

3° Likewise we find

$$\sum \sin a \sin (b-c) \sin (b+c-a) =$$
$$= \frac{1}{2} \{\sin 2(b-a) + \sin 2(c-b) + \sin 2(a-c)\}.$$

It only remains to show that

$$\frac{1}{2} \{\sin 2(b-a) + \sin 2(c-b) + \sin 2(a-c)\} =$$
$$= 2 \sin (b-c) \sin (c-a) \sin (a-b).$$

4° Proved analogously to 3° or by replacing a by $\frac{\pi}{2}-a$, b by $\frac{\pi}{2}-b$ and c by $\frac{\pi}{2}-c$.

54. 1° We have

$$\sum \sin^3 A \cos (B-C) = \sum \sin^2 A \sin A \cos (B-C) =$$
$$= \frac{1}{2} \sum \sin^2 A \{\sin (A+B-C + \sin (A-B+C)\}.$$

But since $A+B+C=\pi$, we have

$$\sum \sin^2 A \cos (B-C) = \frac{1}{2} \sum \sin^2 A (\sin 2C + \sin 2B) =$$
$$= \sum \sin^2 A (\sin B \cos B + \sin C \cos C) =$$
$$= \sin^2 A \sin B \cos B + \sin^2 A \sin C \cos C +$$
$$+ \sin^2 B \sin C \cos C + \sin^2 B \sin A \cos A +$$
$$+ \sin^2 C \sin A \cos A + \sin^2 C \sin B \cos B =$$
$$= \sin A \sin B (\sin A \cos B + \cos A \sin B) +$$
$$+ \sin A \sin C (\sin A \cos C + \cos A \sin C) +$$
$$+ \sin B \sin C (\sin B \cos C + \cos C \sin C) =$$
$$= \sin A \sin B \sin (A+B) + \sin A \sin C \sin (A+C) +$$
$$+ \sin B \sin C \sin (B+C) = 3 \sin A \sin B \sin C.$$

2° We have

$$\sum \sin^3 A \sin(B-C) = \sum \sin^2 A \sin A \sin(B-C) =$$

$$= \sum \sin^2 A \sin(B+C) \sin(B-C) =$$

$$= \frac{1}{2} \sum \sin^2 A \{\cos 2C - \cos 2B\} = \sum \sin^2 A (\sin^2 B - \sin^2 C) =$$

$$= \sin^2 A \sin^2 B \sin^2 C \sum \left(\frac{1}{\sin^2 C} - \frac{1}{\sin^2 B}\right) = \sin^2 A \sin^2 B \sin^2 C \times$$

$$\times \left\{\frac{1}{\sin^2 C} - \frac{1}{\sin^2 B} + \frac{1}{\sin^2 A} - \frac{1}{\sin^2 C} + \frac{1}{\sin^2 B} - \frac{1}{\sin^2 A}\right\} = 0.$$

55. 1° We have

$$\sin 3x = 3 \sin x - 4 \sin^3 x.$$

Therefore

$$\sum \sin 3A \sin^3(B-C) = \frac{1}{4} \sum \sin 3A \{3 \sin(B-C) -$$

$$- \sin 3(B-C)\} = \frac{3}{4} \sum \sin 3(B+C) \sin(B-C) -$$

$$- \frac{1}{4} \sum \sin 3(B+C) \sin 3(B-C) =$$

$$= \frac{3}{8} \sum \{\cos(2B+4C) - \cos(4B+2C)\} -$$

$$- \frac{1}{8} \sum (\cos 6C - \cos 6B) =$$

$$= \frac{3}{8} \{\cos 2(B+2C) - \cos 2(C+2B) + \cos 2(C+2A) -$$

$$- \cos 2(A+2C) + \cos 2(A+2B) - \cos 2(B+2A)\} -$$

$$- \frac{1}{8} \{\cos 6C - \cos 6B + \cos 6A - \cos 6C + \cos 6B - \cos 6A\}.$$

But

$$\cos(2B+4C) = \cos(2B+4A),$$

$$\cos(2C+4B) = \cos(2C+4A),$$

$$\cos(2A+4C) = \cos(2A+4B).$$

And so, we finally have

$$\sum \sin 3A \sin^3 (B - C) = 0.$$

2° Since $\cos 3x = 4 \cos^3 x - 3 \cos x$, we have

$$\sum \sin 3A \cos^3 (B-C) =$$

$$= \frac{1}{4} \sum \sin 3(B+C) \{\cos 3(B-C) + 3\cos(B-C)\} =$$

$$= \frac{1}{4} \sum \sin 3(B+C) \cos 3(B-C) +$$

$$+ \frac{3}{4} \sum \sin 3(B+C) \cos(B-C) =$$

$$= \frac{1}{8} \sum (\sin 6B + \sin 6C) + \frac{3}{8} \sum \{\sin(4B+2C) +$$

$$+ \sin(2B+4C)\} = \frac{1}{4} (\sin 6A + \sin 6B + \sin 6C) =$$

$$= \sin 3A \sin 3B \sin 3C.$$

SOLUTIONS TO SECTION 3

1. The validity of the given identity can be checked, for instance, by the following method. From the formulas (*) (see the beginning of the corresponding section in "Problems") we get

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}, \quad \sqrt{2-\sqrt{3}} = \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}}.$$

Therefore we have

$$\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} = \frac{\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^2}{\sqrt{2}+\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}} = \frac{(1+\sqrt{3})^2 \cdot \sqrt{2}}{2(3+\sqrt{3})} =$$

$$= \frac{(1+\sqrt{3})^2 \cdot \sqrt{2}}{2\sqrt{3}(1+\sqrt{3})} = \frac{1+\sqrt{3}}{\sqrt{6}}$$

Likewise we get

$$\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}=\frac{\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^2}{\sqrt{2}-\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}}=$$

$$=\frac{(1-\sqrt{3})^2\cdot\sqrt{2}}{2(3-\sqrt{3})}=\frac{\sqrt{3}-1}{\sqrt{6}}.$$

Consequently

$$\left(\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}\right)^2=\left(\frac{1+\sqrt{3}}{\sqrt{6}}+\frac{\sqrt{3}-1}{\sqrt{6}}\right)^2=$$

$$=\left(\frac{2\sqrt{3}}{\sqrt{6}}\right)^2=2.$$

2. Let us prove the proposed identities by a direct check.

1° Put $\sqrt[3]{2}=\alpha$, i.e. $\alpha^3=2$. It is required to prove that

$$(1-\alpha+\alpha^2)^3=9(\alpha-1).$$

We have

$$(1-\alpha+\alpha^2)^2=1+\alpha^2+\alpha^4+2\alpha^2-2\alpha^3-2\alpha=$$
$$=3(\alpha^2-1),$$

since

$$\alpha^3=2,\ \alpha^4=2\alpha.$$

Hence

$$(1-\alpha+\alpha^2)^3=3(\alpha^2-\alpha+1)(\alpha^2-1)=$$
$$=3(\alpha^2-\alpha+1)(\alpha+1)(\alpha-1)=$$
$$=3(\alpha^3+1)(\alpha-1)=9(\alpha-1).$$

2° We have to prove that

$$(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25})^2=9(\sqrt[3]{5}-\sqrt[3]{4}).$$

Squaring the left member, we find

$$\sqrt[3]{4}+\sqrt[3]{400}+\sqrt[3]{625}+2\sqrt[3]{40}-2\sqrt[3]{50}-2\sqrt[3]{500}=$$
$$=\sqrt[3]{4}+2\sqrt[3]{50}+5\sqrt[3]{5}+4\sqrt[3]{5}-2\sqrt[3]{50}-10\sqrt[3]{4}=$$
$$=9(\sqrt[3]{5}-\sqrt[3]{4}).$$

3° Proved as in the preceding case.

4° We have to prove that

$$\left(\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right)^4=\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}.$$

Put

$$\sqrt[4]{5}=\alpha.$$

We have

$$\left(\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right)^4=\frac{(\alpha+1)^4}{(\alpha-1)^4}=\frac{1+4\alpha+6\alpha^2+4\alpha^3+\alpha^4}{1-4\alpha+6\alpha^2-4\alpha^3+\alpha^4}=$$

$$=\frac{3+2\alpha+3\alpha^2+2\alpha^3}{3-2\alpha+3\alpha^2-2\alpha^3},$$

since $\alpha^4=5$.

Further

$$\left(\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\right)^4=\frac{3+2\alpha+\alpha^2(3+2\alpha)}{3-2\alpha+\alpha^2(3-2\alpha)}=\frac{3+2\alpha}{3-2\alpha}=\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}.$$

5° It is required to prove that

$$(1+\sqrt[5]{3}-\sqrt[5]{9})^3=5(2-\sqrt[5]{27}).$$

Put

$$\sqrt[5]{3}=\alpha, \text{ i.e. } \alpha^5=3.$$

We have

$$(1+\alpha-\alpha^2)^2=1+\alpha^2+\alpha^4+2\alpha-2\alpha^2-2\alpha^3=$$
$$=1+2\alpha-\alpha^2-2\alpha^3+\alpha^4.$$

Further

$$(1+\alpha-\alpha^2)^3=1+3\alpha-5\alpha^3+3\alpha^5-\alpha^6.$$

But

$$\alpha^6=3\alpha,\ \alpha^5=3.$$

Therefore

$$(1+\alpha-\alpha^2)^3=10-5\alpha^3=5(2-\sqrt[5]{27}).$$

6° Put $\sqrt[5]{2}=\alpha$ and prove the first equality which can be rewritten in the following form

$$5(1+\alpha+\alpha^3)^2=(1+\alpha^2)^5.$$

The right member is equal to

$$1 + 5\alpha^2 + 10\alpha^4 + 10\alpha^6 + 5\alpha^8 + \alpha^{10} =$$
$$= 5(1 + \alpha^2 + 2\alpha^4 + 2\alpha^6 + \alpha^8),$$

since

$$\alpha^{10} = 4.$$

Further

$$\alpha^5 = 2, \quad \alpha^6 = 2\alpha, \quad \alpha^8 = 2\alpha^3,$$

and, consequently,

$$(1 + \alpha^2)^5 = 5(1 + \alpha^2 + 2\alpha^4 + 4\alpha + 2\alpha^3).$$

It only remains to prove that

$$(1 + \alpha + \alpha^3)^2 = 1 + 4\alpha + \alpha^2 + 2\alpha^3 + 2\alpha^4.$$

The last equality is readily proved by removing the brackets in the left member and performing simple transformations. To prove the second equality we have to show that

$$\sqrt[5]{\frac{1}{5}} + \sqrt[5]{\frac{4}{5}} = \left(\sqrt[5]{\frac{16}{125}} + \sqrt[5]{\frac{8}{125}} + \sqrt[5]{\frac{2}{125}} - \sqrt[5]{\frac{1}{125}}\right)^2,$$

or

$$5(1 + \sqrt[5]{4}) = (\sqrt[5]{16} + \sqrt[5]{8} + \sqrt[5]{2} - 1)^2.$$

Put

$$\sqrt[5]{2} = \alpha, \quad \alpha^5 = 2, \quad \alpha^6 = 2\alpha, \quad \alpha^7 = 2\alpha^2, \quad \alpha^8 = 2\alpha^3.$$

Then we have to prove that

$$(\alpha^4 + \alpha^3 + \alpha - 1)^2 = 5(1 + \alpha^2).$$

Expanding the left member, we find

$$1 + \alpha^2 + \alpha^6 + \alpha^8 + 2\alpha^7 + 2\alpha^5 - 2\alpha^4 + 2\alpha^4 - 2\alpha^3 - 2\alpha.$$

Making use of the equalities enabling us to replace high powers of α by lower ones, we find the required identity.

3. Put

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{D}{d} = \lambda.$$

Then

$$A = a\lambda, \quad B = b\lambda, \quad C = c\lambda, \quad D = d\lambda.$$

Consequently

$$\sqrt{Aa}+\sqrt{Bb}+\sqrt{Cc}+\sqrt{Dd}=\sqrt{\lambda}\,(a+b+c+d).$$

But

$$A+B+C+D=\lambda\,(a+b+c+d)$$

and

$$\lambda=\frac{A+B+C+D}{a+b+c+d},$$

i.e.

$$\sqrt{\lambda}=\frac{\sqrt{A+B+C+D}}{\sqrt{a+b+c+d}}.$$

Replacing $\sqrt{\lambda}$ in the equality

$$\sqrt{Aa}+\sqrt{Bb}+\sqrt{Cc}+\sqrt{Dd}=\sqrt{\lambda}\,(a+b+c+d)$$

by the found value, we obtain the required identity.

4. Put for brevity

$$\sqrt[3]{ax^2+by^2+cz^2}=A.$$

We have

$$A=\sqrt[3]{\frac{ax^3}{x}+\frac{by^3}{y}+\frac{cz^3}{z}}=\sqrt[3]{ax^3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}=x\sqrt[3]{a},$$

since

$$ax^3=by^3=cz^3 \quad \text{and} \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

Likewise we find

$$A=y\sqrt[3]{b} \quad \text{and} \quad A=z\sqrt[3]{c}.$$

Hence

$$\frac{A}{x}=\sqrt[3]{a},\quad \frac{A}{y}=\sqrt[3]{b},\quad \frac{A}{z}=\sqrt[3]{c}.$$

Adding these equalities termwise, we get

$$A\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}.$$

Hence, finally,

$$A=\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}.$$

5. Put

$$1+\frac{1}{\sqrt{2}}=\alpha, \quad 1-\frac{1}{\sqrt{2}}=\beta.$$

Then

$$a_n=\alpha^n+\beta^n, \quad b_n=\alpha^n-\beta^n,$$

where $\alpha\beta=\frac{1}{2}$.

Prove that

$$a_m a_n-\frac{a_{m-n}}{2^n}=a_{m+n}.$$

We have

$$a_m a_n-\frac{a_{m-n}}{2^n}=(\alpha^m+\beta^m)(\alpha^n+\beta^n)-\frac{\alpha^{m-n}+\beta^{m-n}}{2^n}=$$
$$=\alpha^{m+n}+\beta^{m+n}+\alpha^n\beta^n(\alpha^{m-n}+\beta^{m-n})-$$
$$-\frac{\alpha^{m-n}+\beta^{m-n}}{2^n}.$$

But

$$\alpha^n\beta^n=\frac{1}{2^n},$$

consequently,

$$a_m a_n-\frac{a_{m-n}}{2^n}=\alpha^{m+n}+\beta^{m+n}=a_{m+n}.$$

The second relation is proved in the same way.

6. Put

$$\frac{1+\sqrt{5}}{2}=\alpha, \quad \frac{1-\sqrt{5}}{2}=\beta.$$

Then

$$\alpha+\beta=1, \quad \alpha\beta=-1.$$

Furthermore

$$\alpha^2-\alpha-1=0, \quad \beta^2-\beta-1=0$$

and

$$u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n).$$

Proof. 1° We have

$$u_n + u_{n-1} = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) + \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1}) =$$
$$= \frac{1}{\sqrt{5}}\{(\alpha^n + \alpha^{n-1}) - (\beta^n + \beta^{n-1})\}.$$

Multiplying both members of the equality $\alpha^2 - \alpha - 1 = 0$ by α^{n-1}, we get

$$\alpha + 1 = \alpha^2, \quad \alpha^n + \alpha^{n-1} = \alpha^{n+1}.$$

Analogously, it is easy to conclude that

$$\beta^n + \beta^{n-1} = \beta^{n+1}.$$

Therefore

$$u_n + u_{n-1} = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}) = u_{n+1}.$$

2° We have

$$u_k u_{n-k} + u_{k-1} u_{n-k-1} =$$
$$= \frac{1}{5}\{(\alpha^k - \beta^k)(\alpha^{n-k} - \beta^{n-k}) + (\alpha^{k-1} - \beta^{k-1})(\alpha^{n-k-1} - \beta^{n-k-1})\} =$$
$$= \frac{1}{5}\{\alpha^n + \beta^n - \alpha^k\beta^{n-k} - \beta^k\alpha^{n-k} + \alpha^{n-2} + \beta^{n-2} - \beta^{k-1}\alpha^{n-k-1} -$$
$$- \beta^{n-k-1}\alpha^{k-1}\} =$$
$$= \frac{1}{5}\left\{\alpha^n + \alpha^{n-2} + \beta^n + \beta^{n-2} - \beta^n\left(\frac{\alpha^k}{\beta^k} + \frac{\alpha^{k-1}}{\beta^{k+1}}\right) -\right.$$
$$\left. - \alpha^n\left(\frac{\beta^k}{\alpha^k} + \frac{\beta^{k-1}}{\alpha^{k+1}}\right)\right\} =$$
$$= \frac{1}{5}\left\{\alpha^n + \alpha^{n-2} + \beta^n + \beta^{n-2} - \beta^n\frac{\alpha^k\beta + \alpha^{k-1}}{\beta^{k+1}} - \alpha^n\frac{\beta^k\alpha + \beta^{k-1}}{\alpha^{k+1}}\right\} =$$
$$= \frac{1}{5}\left\{\alpha^n + \alpha^{n-2} + \beta^n + \beta^{n-2} - \beta^n\frac{\alpha^{k-1}(\alpha\beta + 1)}{\beta^{k+1}} - \alpha^n\frac{\beta^{k-1}(\alpha\beta + 1)}{\alpha^{k+1}}\right\} =$$
$$= \frac{1}{5}\{\alpha^n + \alpha^{n-2} + \beta^n + \beta^{n-2}\},$$

since $\alpha\beta+1=0$. Then we perform the following transformations

$$\frac{1}{5}\{\alpha^n+\alpha^{n-2}+\beta^n+\beta^{n-2}\}=\frac{1}{5}\left\{\alpha^{n-1}\left(\alpha+\frac{1}{\alpha}\right)+\beta^{n-1}\left(\beta+\frac{1}{\beta}\right)\right\}=$$

$$=\frac{1}{5}\{\alpha^{n-1}(\alpha-\beta)+\beta^{n-1}(\beta-\alpha)\}=\frac{\alpha-\beta}{5}(\alpha^{n-1}-\beta^{n-1})=$$

$$=\frac{1}{\sqrt{5}}(\alpha^{n-1}-\beta^{n-1})=u_{n-1}.$$

3° Obtained from 2° by putting $n=2k$, and then replacing k by n.

4° We have to show that

$$5(\alpha^{3n}-\beta^{3n})-(\alpha^n-\beta^n)^3-(\alpha^{n+1}-\beta^{n+1})^3+(\alpha^{n-1}-\beta^{n-1})^3=0.$$

The left member is transformed in the following way

$$5(\alpha^{3n}-\beta^{3n})-\alpha^{3n}\left(\alpha^3+1-\frac{1}{\alpha^3}\right)+3\alpha^{2n}\beta^n\left(\alpha^2\beta+1-\frac{1}{\alpha^2\beta}\right)-$$

$$-3\alpha^n\beta^{2n}\left(\alpha\beta^2+1-\frac{1}{\alpha\beta^2}\right)+\beta^{3n}\left(\beta^3+1-\frac{1}{\beta^3}\right).$$

It is easy to show that $\alpha^2\beta+1-\frac{1}{\alpha^2\beta}=0$, $\alpha\beta^2+1-\frac{1}{\alpha\beta^2}=0$. On the other hand, we can easily make sure that

$$\alpha^3+1-\frac{1}{\alpha^3}=\beta^3+1-\frac{1}{\beta^3}=\alpha^3+\beta^3+1=$$

$$=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)+1=\alpha^2-\alpha\beta+\beta^2+1=5.$$

Hence follows the validity of our identity.

5° We have to prove that

$$(\alpha^n-\beta^n)^4-(\alpha^{n-2}-\beta^{n-2})(\alpha^{n-1}-\beta^{n-1})(\alpha^{n+1}-\beta^{n+1})\times$$

$$\times(\alpha^{n+2}-\beta^{n+2})=25.$$

First prove that

$$(\alpha^{n-2}-\beta^{n-2})(\alpha^{n+2}-\beta^{n+2})=\alpha^{2n}+\beta^{2n}-(-1)^n(\alpha^4+\beta^4),$$

$$(\alpha^{n-1}-\beta^{n-1})(\alpha^{n+1}-\beta^{n+1})=\alpha^{2n}+\beta^{2n}+(-1)^n(\alpha^2+\beta^2).$$

But

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=3,\quad \alpha^4+\beta^4=$$

$$=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=7.$$

Therefore

$$(\alpha^{n-2}-\beta^{n-2})(\alpha^{n-1}-\beta^{n-1})(\alpha^{n+1}-\beta^{n+1})(\alpha^{n+2}-\beta^{n+2})=$$
$$=(\alpha^{2n}+\beta^{2n})^2-(-1)^n 4(\alpha^{2n}+\beta^{2n})-21.$$

On the other hand

$$(\alpha^n-\beta^n)^4=\alpha^{4n}-4\alpha^{3n}\beta^n+4-4\alpha^n\beta^{3n}+\beta^{4n}=$$
$$=\alpha^{4n}+\beta^{4n}+4-4(-1)^n(\alpha^{2n}+\beta^{2n}).$$

Subtracting the last-but-one equality from the last one termwise, we find the required result.

6° and 7° are proved analogously to the previous cases.

7. 1° We have

$$2\left[(a^2+b^2)^{\frac{1}{2}}-a\right]\left[(a^2+b^2)^{\frac{1}{2}}-b\right]=$$
$$=2(a^2+b^2)-2(a+b)(a^2+b^2)^{\frac{1}{2}}+2ab=$$
$$=(a^2+b^2)-2(a+b)\sqrt{a^2+b^2}+(a+b)^2+$$
$$+(a^2+b^2)+2ab-(a+b)^2$$

(singling out a perfect square).

Consequently

$$2\left[(a^2+b^2)^{\frac{1}{2}}-a\right]\left[(a^2+b^2)^{\frac{1}{2}}-b\right]=\left(a+b-\sqrt{a^2+b^2}\right)^2.$$

Hence follows the first identity.

2° Multiplying the braced expressions on the left, we get

$$3(a^3+b^3)^{\frac{2}{3}}-3(a+b)(a^3+b^3)^{\frac{1}{3}}+3ab=$$
$$=3(a^2-ab+b^2)^{\frac{2}{3}}(a+b)^{\frac{2}{3}}-3(a^2-ab+b^2)^{\frac{1}{3}}(a+b)^{\frac{4}{3}}+$$
$$+(a+b)^2-(a^2-ab+b^2)=\left[(a+b)^{\frac{2}{3}}-(a^2-ab+b^2)^{\frac{1}{3}}\right]^3.$$

The rest is obvious.

8. It is easily seen that $ax=\sqrt{\frac{2a-b}{b}}$, hence

$$\frac{1-ax}{1+ax}=\frac{1-\sqrt{\frac{2a-b}{b}}}{1+\sqrt{\frac{2a-b}{b}}}=\frac{\left(1-\sqrt{\frac{2a-b}{b}}\right)^2}{1-\frac{2a-b}{b}}=$$

$$=\frac{b}{2(b-a)}\left(1-2\sqrt{\frac{2a-b}{b}}+\frac{2a-b}{b}\right)=\frac{a-b\sqrt{\frac{2a-b}{b}}}{b-a}.$$

Analogously, we find

$$\sqrt{\frac{1+bx}{1-bx}}=\sqrt{\frac{1+\frac{b}{a}\sqrt{\frac{2a-b}{b}}}{1-\frac{b}{a}\sqrt{\frac{2a-b}{b}}}}=\frac{1+\frac{b}{a}\sqrt{\frac{2a-b}{b}}}{\sqrt{1-\frac{b^2}{a^2}\cdot\frac{2a-b}{b}}}=$$

$$=\frac{a+b\sqrt{\frac{2a-b}{b}}}{\sqrt{a^2-2ab+b^2}}=\frac{a+b\sqrt{\frac{2a-b}{b}}}{\sqrt{(b-a)^2}}=\frac{a+b\sqrt{\frac{2a-b}{b}}}{b-a}$$

(since $b-a>0$). Multiplying the two obtained expressions, we find

$$\frac{a-b\sqrt{\frac{2a-b}{b}}}{b-a}\cdot\frac{a+b\sqrt{\frac{2a-b}{b}}}{b-a}=\frac{a^2-b^2\frac{2a-b}{b}}{(b-a)^2}=$$

$$=\frac{a^2-2ab+b^2}{(b-a)^2}=1.$$

9. Factor the expression

$$n^3-3n-2.$$

We have

$$n^3-3n-2=n^3-n-2n-2=n(n^2-1)-$$
$$-2(n+1)=(n+1)(n^2-n-2)=$$
$$=(n+1)^2(n-2).$$

Likewise

$$n^3-3n+2=(n-1)^2(n+2).$$

Now we may write:

$$\frac{n^3-3n-2+(n^2-1)\sqrt{n^2-4}}{n^3-3n+2+(n^2-1)\sqrt{n^2-4}}=$$

$$=\frac{(n+1)^2(n-2)+(n^2-1)\sqrt{n^2-4}}{(n-1)^2(n+2)+(n^2-1)\sqrt{n^2-4}}=\frac{(n+1)\sqrt{n-2}}{(n-1)\sqrt{n+2}}\times$$

$$\times\frac{(n+1)\sqrt{n-2}+(n-1)\sqrt{n+2}}{(n-1)\sqrt{n+2}+(n+1)\sqrt{n-2}}=\frac{(n+1)\sqrt{n-2}}{(n-1)\sqrt{n+2}}.$$

10. Consider the second one of the fractions contained in the first brackets, namely:

$$\frac{1-a}{\sqrt{1-a^2}-1+a}=\frac{1-a}{\sqrt{1-a^2}-(1-a)}=\frac{\sqrt{1-a}}{\sqrt{1+a}-\sqrt{1-a}}.$$

And so, the transformed expression takes the form

$$\left[\frac{\sqrt{1+a}}{\sqrt{1+a}-\sqrt{1-a}}+\frac{\sqrt{1-a}}{\sqrt{1+a}-\sqrt{1-a}}\right]\cdot\frac{\sqrt{1-a^2}-1}{a}=$$

$$=\frac{\sqrt{1+a}+\sqrt{1-a}}{\sqrt{1+a}-\sqrt{1-a}}\cdot\frac{\sqrt{1-a^2}-1}{a}=$$

$$=\frac{2a}{(\sqrt{1+a}-\sqrt{1-a})^2}\cdot\frac{(\sqrt{1-a^2}-1)}{a}=$$

$$=\frac{2(\sqrt{1-a^2}-1)}{(1+a+1-a-2\sqrt{1-a^2})}=-1.$$

11. From the formula (*) it is easy to get:

$$\sqrt{A+\sqrt{B}}+\sqrt{A-\sqrt{B}}=2\sqrt{\frac{A+\sqrt{A^2-B}}{2}}.$$

In our case

$$A=x,\quad B=4x-4,\quad A^2-B=x^2-4x+4,$$

$$\sqrt{A^2-B}=\sqrt{(x-2)^2}=\begin{cases}x-2 & \text{if } x>2,\\ 2-x & \text{if } x<2.\end{cases}$$

In the first case we have

$$\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}=2\sqrt{\frac{x+x-2}{2}}=$$

$$=2\sqrt{x-1}.$$

The second case yields

$$\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}=2\sqrt{\frac{x+2-x}{2}}=2.$$

It is easy to see that at $x = 2$ the expression under consideration is also equal to 2.

12. In this case

$$A = a + b + c, \quad B = 4ac + 4bc,$$

$$A^2 - B = (a + b + c)^2 - 4ac - 4bc =$$
$$= a^2 + b^2 + c^2 + 2ab - 2bc - 2ac =$$
$$= (a + b - c)^2.$$

If

$$a + b - c > 0,$$

then

$$\sqrt{A^2 - B} = a + b - c.$$

If

$$a + b - c < 0,$$

then

$$\sqrt{A^2 - B} = c - a - b.$$

Hence, we easily obtain that the given expression is equal to $2\sqrt{a+b}$ if $a + b > c$, and to $2\sqrt{c}$ if $a + b < c$. At $a + b = c$ these values coincide.

13. Let us denote

$$\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}=u, \quad \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}=v.$$

Then

$$x = u + v.$$

Consequently

$$x^3 = (u + v)^3 = u^3 + v^3 + 3uv(u + v).$$

But

$$u^3 + v^3 = -q, \quad uv = -\frac{p}{3}.$$

Therefore

$$x^3 = -q - px$$

or

$$x^3 + px + q = 0$$

which is the required result.

14. We can proceed, for instance, in the following way. Put

$$\sqrt{x+a}+\sqrt{x+b}=z.$$

Then (multiplying and dividing the left member by $\sqrt{x+a}-$ $-\sqrt{x+b}$) we find.

$$\frac{a-b}{\sqrt{x+a}-\sqrt{x+b}}=z$$

or

$$\sqrt{x+a}-\sqrt{x+b}=\frac{a-b}{z}.$$

Hence

$$2\sqrt{x+a}=z+\frac{a-b}{z}, \quad 2\sqrt{x+b}=z-\frac{a-b}{z},$$

i.e. both roots are expressed in terms of z without radicals.

15. Put

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{1}{\lambda}.$$

Consequently

$$a'=a\lambda, \quad b'=b\lambda, \quad c'=c\lambda, \quad \lambda=\frac{a'+b'+c'}{a+b+c}.$$

Therefore

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a'}+\sqrt{b'}+\sqrt{c'}=$$
$$=(1+\sqrt{\lambda})(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Our fraction takes the form

$$\frac{1}{(1+\sqrt{\lambda})(\sqrt{a}+\sqrt{b}+\sqrt{c})}=\frac{(1-\sqrt{\lambda})(\sqrt{a}+\sqrt{b}-\sqrt{c})}{(1-\lambda)(a+b-c+2\sqrt{ab})}=$$
$$=\frac{(1-\sqrt{\lambda})(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})}{(1-\lambda)(a^2+b^2+c^2-2ab-2ac-2bc)}=$$
$$=\frac{(\sqrt{a+b+c}-\sqrt{a'+b'+c'})(\sqrt{a}+\sqrt{b}-\sqrt{c}+)(a+b-c-2\sqrt{ab})\sqrt{a+b+c}}{(a+b+c-a'-b'-c')(a^2+b^2+c^2-2ab-2ac-2bc)}.$$

16. Put

$$\sqrt[3]{2}=p+\sqrt{q}.$$

Hence

$$2=p^3+3pq+(3p^2+q)\sqrt{q},$$

since q is not a perfect square, it must be $3p^2 + q = 0$, which is impossible.

17. 1° We have

$$\tan\left(\frac{3\pi}{2}-\alpha\right)=\tan\left(\pi+\frac{\pi}{2}-\alpha\right)=\tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha,$$

$$\cos\left(\frac{3\pi}{2}-\alpha\right)=\cos\left(\pi+\frac{\pi}{2}-\alpha\right)=-\cos\left(\frac{\pi}{2}-\alpha\right)=$$

$$=-\sin\alpha \quad (2°,\ 4°),$$

$$\cos(2\pi-\alpha)=\cos(-\alpha)=\cos\alpha \qquad (1°,\ 3°),$$

$$\cos\left(\alpha-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha \qquad (3°,\ 4°),$$

$$\sin(\pi-\alpha)=-\sin(-\alpha)=+\sin\alpha \qquad (2°,\ 3°),$$

$$\cos(\pi+\alpha)=-\cos\alpha \qquad (2°),$$

$$\sin\left(\alpha-\frac{\pi}{2}\right)=-\sin\left(\frac{\pi}{2}-\alpha\right)=-\cos\alpha \qquad (3°,\ 4°).$$

Now we get

$$\frac{-\cot\alpha\cdot\sin\alpha}{\cos\alpha}+\sin^2\alpha+\cos^2\alpha=-1+\sin^2\alpha+\cos^2\alpha=0.$$

2° In this case we obtain

$$\sin(3\pi-\alpha)=(-1)^3\sin(-\alpha)=-\sin(-\alpha)=\sin\alpha \quad (2°,\ 3°),$$

$$\cos(3\pi+\alpha)=(-1)^3\cos\alpha=-\cos\alpha \qquad (2°),$$

$$\sin\left(\frac{3\pi}{2}-\alpha\right)=\sin\left(\pi+\frac{\pi}{2}-\alpha\right)=-\sin\left(\frac{\pi}{2}-\alpha\right)=$$

$$=-\cos\alpha \quad (2°,\ 4°),$$

$$\cos\left(\frac{5\pi}{2}-\alpha\right)=\cos\left(2\pi+\frac{\pi}{2}-\alpha\right)=\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$$

$$(1° \text{ or } 2°,\ 4°).$$

Thus, we have

$$(1-\sin\alpha-\cos\alpha)(1+\cos\alpha+\sin\alpha)+\sin 2\alpha=$$
$$=[1-(\sin\alpha+\cos\alpha)][1+(\sin\alpha+\cos\alpha)]+\sin 2\alpha=$$
$$=1-(\sin\alpha+\cos\alpha)^2+\sin 2\alpha=$$
$$=1-\sin^2\alpha-\cos^2\alpha-2\sin\alpha\cos\alpha+\sin 2\alpha=0.$$

3° Analogous to the previous ones.

18. Indeed, we have

$$1-\cos\alpha=2\sin^2\frac{\alpha}{2},$$

whence

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

But in our conditions

$$\frac{\alpha}{2}=k\pi+\frac{\alpha_0}{2}\quad\left(0\leqslant\frac{\alpha_0}{2}<\pi\right).$$

Then

$$\sin\frac{\alpha}{2}=\sin\left(k\pi+\frac{\alpha_0}{2}\right)=(-1)^k\sin\frac{\alpha_0}{2},$$

where

$$\sin\frac{\alpha_0}{2}\geqslant 0.$$

Therefore, indeed

$$\sin\frac{\alpha}{2}=(-1)^k\sqrt{\frac{1-\cos\alpha}{2}}.$$

The second assertion is proved analogously.

19. Let us prove the validity of some of the proposed formulas. Let us, for instance, prove that $A_{16}=0$ if $n\equiv 0 \pmod 2$. Put $n=2l$. Then

$$\frac{1}{2}A_{16}=\cos\left(\frac{l\pi}{4}+\pi-\frac{3}{32}\pi\right)+\cos\left(\frac{3l\pi}{4}+\pi-\frac{5}{32}\pi\right)+$$
$$+\cos\left(\frac{5l\pi}{4}+\frac{5}{32}\pi\right)+\cos\left(\frac{7l\pi}{4}+\frac{3}{32}\pi\right)=$$
$$=-\cos\left(\frac{l\pi}{4}-\frac{3}{32}\pi\right)-\cos\left(l\pi-\frac{l\pi}{4}-\frac{5}{32}\pi\right)+$$
$$+\cos\left(\frac{l\pi}{4}+l\pi+\frac{5}{32}\pi\right)+\cos\left(2l\pi-\frac{l\pi}{4}+\frac{3}{32}\pi\right)=$$
$$=-\cos\left(\frac{l\pi}{4}-\frac{3}{32}\pi\right)-(-1)^l\cos\left(\frac{l\pi}{4}+\frac{5}{32}\pi\right)+$$
$$+(-1)^l\cos\left(\frac{l\pi}{4}+\frac{5}{32}\pi\right)+\cos\left(\frac{l\pi}{4}-\frac{3}{32}\pi\right)=0.$$

Let us prove, for instance, that $A_{14} = 0$ if $n \equiv 1, 3, 4 \pmod 7$. We have:

$$\frac{1}{2} A_{14} = \cos\left(\frac{1}{7} n\pi - \frac{13}{14}\pi\right) + \cos\left(\frac{3}{7} n\pi - \frac{3}{14}\pi\right) + \\ + \cos\left(\frac{5}{7} n\pi - \frac{3}{14}\pi\right).$$

If we replace here n by a number, which is comparable with it by modulus 7, then all the cosines will acquire only a common factor equal to ± 1. Indeed, let us assume that $n \equiv \alpha \pmod 7$, i.e. $n = \alpha + 7N$, where N is an integer. Therefore

$$\cos\left(\frac{kn\pi}{7} - \beta\right) = \cos\left(\frac{k(\alpha + 7N)\pi}{7} - \beta\right) = \\ = \cos\left(\frac{k\alpha\pi}{7} + kN\pi - \beta\right) = (-1)^{kN} \cos\left(\frac{k\alpha\pi}{7} - \beta\right) = \\ = (-1)^{N} \cos\left(\frac{k\alpha\pi}{7} - \beta\right),$$

since in our case $k = 1, 3, 5$ and, consequently, is odd; (β is equal either to $\frac{3}{14}\pi$ or to $\frac{13}{14}\pi$). Therefore, in order to prove that $A_{14} = 0$ at $n \equiv 1, 3, 4 \pmod 7$, it is sufficient to prove that it will take place at $n = 1, 3, 4$. The validity of this is readily checked.

First put $n = 1$. Then we prove that

$$\cos\left(\frac{1}{7}\pi - \frac{13}{14}\pi\right) + \cos\left(\frac{3}{7}\pi - \frac{3}{14}\pi\right) + \cos\left(\frac{5}{7}\pi - \frac{3}{14}\pi\right) = 0.$$

After transformations we get:

$$\cos\frac{11}{14}\pi + \cos\frac{3}{14}\pi + \cos\frac{7}{14}\pi = \cos\left(\pi - \frac{3}{14}\pi\right) + \cos\frac{3}{14}\pi + \\ + \cos\frac{\pi}{2} = -\cos\frac{3}{14}\pi + \cos\frac{3}{14}\pi = 0.$$

Let now $n = 3$. Then we have to prove that

$$\cos\left(\frac{3}{7}\pi - \frac{13}{14}\pi\right) + \cos\left(\frac{9}{7}\pi - \frac{3}{14}\pi\right) + \cos\left(\frac{15}{7}\pi - \frac{3}{14}\pi\right) = \\ = \cos\frac{7}{14}\pi + \cos\frac{15}{14}\pi + \cos\frac{27}{14}\pi = \cos\left(\pi + \frac{\pi}{14}\right) + \\ + \cos\left(2\pi - \frac{\pi}{14}\right) = -\cos\frac{\pi}{14} + \cos\frac{\pi}{14} = 0.$$

Reasoning in the same way, we make sure that at $n = 4$ we also obtain zero.

In conclusion, let us prove that A_8 never becomes zero, i.e. at no whole values of n. We have

$$\frac{1}{2}A_8=\cos\left(\frac{1}{4}n\pi-\frac{7}{16}\pi\right)+\cos\left(-\frac{1}{4}n\pi+n\pi-\frac{1}{16}\pi\right)=$$
$$=\cos\left(\frac{1}{4}n\pi-\frac{7}{16}\pi\right)+(-1)^n\cos\left(\frac{1}{4}n\pi+\frac{1}{16}\pi\right).$$

Consider the following cases:

1° Let $n\equiv 0 \pmod 4$, $n = 4N$. Then

$$\frac{1}{2}A_8=\cos\left(N\pi-\frac{7}{16}\pi\right)+(-1)^{4N}\cos\left(N\pi+\frac{1}{16}\pi\right)=$$
$$=(-1)^N\cos\frac{7}{16}\pi+(-1)^N\cos\frac{1}{16}\pi=$$
$$=(-1)^N\left(\cos\frac{1}{16}\pi+\cos\frac{7}{16}\pi\right).$$

The bracketed expression is not equal to zero, since it represents a sum of cosines of two acute angles.

2° Let $n\equiv 1 \pmod 4$, i.e. $n = 1 + 4N$.

$$\frac{1}{2}A_8=\cos\left(\frac{\pi}{4}+N\pi-\frac{7}{16}\pi\right)+\cos\left(\frac{3\pi}{4}+3N\pi-\frac{1}{16}\pi\right)=$$
$$=(-1)^N\left\{\cos\left(\frac{\pi}{4}-\frac{7}{16}\pi\right)+\cos\left(\frac{3\pi}{4}-\frac{1}{16}\pi\right)\right\}=$$
$$=(-1)^N\left\{\cos\frac{3}{16}\pi+\cos\frac{1}{16}\pi\right\}.$$

It is obvious that the braced sum is not equal to zero, and, consequently, in this case A_8 is also not equal to zero. It only remains to consider the cases: $n\equiv 3 \pmod 4$ and $n\equiv 2 \pmod 4$, but we leave them to the reader.

20. It is required to prove that

$$\sum p(k) = 0$$

if $k = n,\ n-1,\ n-2,\ n-4,\ n-5,\ n-6$, and the sign before $p(k)$ is chosen accordingly.

It is evident that

$$\sum p(k)=A\sum(k+3)^2+C\sum(-1)^k+D\sum\cos\frac{2\pi k}{3}.$$

The first two sums on the right are equal to zero. It remains to prove that

$$\sum \cos \frac{2\pi k}{3} = 0.$$

If k is a whole number, the following cases are possible:

1° k is exactly divisible by 3, $k = 3l$;

2° k, when divided by 3, leaves the remainder 1, $k = 3l + 1$;

3° k, when divided by 3, leaves the remainder 2, $k = 3l+2$.

In case 1°

$$\cos \frac{2\pi k}{3} = 1.$$

In cases 2° and 3° $\cos \frac{2\pi k}{3} = \cos \frac{2\pi}{3}$.

Let us first assume that n is divisible by 3. Then

$$\sum \cos \frac{2\pi k}{3} = \frac{\cos 2\pi n}{3} - \cos \frac{2\pi (n-1)}{3} - \cos \frac{2\pi (n-2)}{3} +$$

$$+ \cos \frac{2\pi (n-4)}{3} + \cos \frac{2\pi (n-5)}{3} - \cos \frac{2\pi (n-6)}{3}.$$

But

$$2 \equiv -1 \pmod 3$$

and

$$\cos \frac{2\pi k}{3} = \cos \frac{2\pi k'}{3}$$

if

$$k \equiv k' \pmod 3).$$

Since by the assumption $n \equiv 0 \pmod 3$, we have

$n - 1 \equiv -1$, $n - 2 \equiv 1$, $n - 4 \equiv -1$,

$n - 5 \equiv +1$, $n - 6 \equiv 0$,

and our sum takes the form

$$1 - \cos \frac{2\pi}{3} - \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} - 1 = 0.$$

It remains to prove that our sum is also equal to zero in the cases when $n \equiv \pm 1 \pmod 3$. The proof is similar to the previous case.

21. We have

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Analogously we find cos 15°.

We have

$$\sin 18^\circ = \sin\frac{\pi}{10} = \cos\frac{2\pi}{5}.$$

But

$$2\sin\frac{\pi}{5}\cos\frac{\pi}{5} = \sin\frac{2\pi}{5},$$

$$2\sin\frac{2\pi}{5}\cos\frac{2\pi}{5} = \sin\frac{4\pi}{5} = \sin\frac{\pi}{5}.$$

Multiplying these equalities termwise, we find

$$\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}.$$

On the other hand

$$\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = 2\sin\frac{3\pi}{10}\sin\frac{\pi}{10} = 2\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{2}.$$

Thus, if we put

$$\sin\frac{\pi}{10} = \cos\frac{2\pi}{5} = x, \quad \cos\frac{\pi}{5} = y,$$

we have

$$y - x = \frac{1}{2}, \quad xy = \frac{1}{4}.$$

But

$$(x+y)^2 = (x-y)^2 + 4xy = \frac{1}{4} + 1 = \frac{5}{4}.$$

Consequently,

$$x + y = \frac{\sqrt{5}}{2}.$$

Using this relation and the relation $y - x = \frac{1}{2}$, we get

$$x = \sin\frac{\pi}{10} = \sin 18^\circ = \frac{-1+\sqrt{5}}{4}.$$

Now cos 18° is readily found.

22. Indeed

$$\sin 6^\circ = \sin(60^\circ - 54^\circ) = \sin 60^\circ \cos 54^\circ - \cos 60^\circ \sin 54^\circ.$$

But

$$\sin 54^\circ = \cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\frac{6-2\sqrt{5}}{16} = \frac{1+\sqrt{5}}{4},$$

$$\cos 54^\circ = \sqrt{1-\sin^2 54^\circ} = \frac{1}{4}\sqrt{10-2\sqrt{5}}.$$

To obtain the result we have to substitute these values into the first formula; $\cos 6^\circ$ is found in the same way.

23. Bear in mind that

$$(1)\quad -\frac{\pi}{2} \leqslant \arcsin x \leqslant +\frac{\pi}{2}, \qquad -\frac{\pi}{2} < \arctan x < +\frac{\pi}{2},$$

$$0 \leqslant \arccos x \leqslant \pi, \qquad 0 < \operatorname{arccot} x < \pi,$$

$$(2)\quad \sin(\arcsin x) = x, \quad \cos(\arccos x) = x,$$

$$\tan(\arctan x) = x, \quad \cot(\operatorname{arccot} x) = x.$$

Let us now prove that

$$\cos(\arcsin x) = \sqrt{1-x^2}.$$

Put

$$\arcsin x = y,$$

then

$$\sin y = x.$$

We have got to compute $\cos y$. But it is known that

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2},$$

and the radical is taken with the plus sign, since

$$-\frac{\pi}{2} \leqslant y \leqslant +\frac{\pi}{2},$$

and, consequently,

$$\cos y \geqslant 0.$$

Let us, for example, also prove that

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

Put

$$\arctan x = y, \quad \tan y = x.$$

We have to find $\cos y$. We have

$$\frac{1}{\cos^2 y} = 1 + \tan^2 y = 1 + x^2.$$

Consequently

$$\cos^2 y = \frac{1}{1+x^2}$$

and

$$\cos y = \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}},$$

where the radical is taken with the plus sign again, since

$$\cos y \geqslant 0.$$

The rest of the formulas are proved in the same way.

24. By definition,

$$-\frac{\pi}{2} < \arctan x < +\frac{\pi}{2},$$

$$0 < \operatorname{arccot} x < \pi.$$

Therefore

$$-\frac{\pi}{2} < \arctan x + \operatorname{arccot} x < +\frac{3\pi}{2}.$$

Let us compute $\sin(\arctan x + \operatorname{arccot} x)$. We have

$$\sin(\arctan x + \operatorname{arccot} x) =$$

$$= \sin(\arctan x)\cos(\operatorname{arccot} x) +$$

$$+ \cos(\arctan x)\sin(\operatorname{arccot} x) =$$

$$= \frac{x}{\sqrt{1+x^2}} \cdot \frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} = 1.$$

However, if the sine of a certain arc is equal to 1, then this arc equals

$$\frac{\pi}{2} + 2k\pi,$$

where k is any whole number, i.e., in other words,

$$\arctan x + \operatorname{arccot} x$$

can attain one of the following values

$$\ldots, \frac{-7\pi}{2}, \frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots.$$

But only one of them, namely $\frac{\pi}{2}$, is contained in the interval between $-\frac{\pi}{2}$ and $+\frac{3\pi}{2}$. Therefore it is obligatory that

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}.$$

Likewise, let us prove that

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

First of all we have

$$-\frac{\pi}{2} \leqslant \arcsin x + \arccos x \leqslant \frac{3\pi}{2}.$$

On the other hand,

$$\sin(\arcsin x + \arccos x) = \sin(\arcsin x)\cos(\arccos x) + \cos(\arcsin x)\sin(\arccos x) = x^2 + \sqrt{1-x^2}\cdot\sqrt{1-x^2} = 1,$$

wherefrom follows that

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

25. First of all it is easy to prove that the quantities

$$\arctan x + \arctan y$$

and

$$\arctan \frac{x+y}{1-xy}$$

differ from each other only by $\varepsilon\pi$, where ε is an integer. Indeed,

$$\tan\left(\arctan \frac{x+y}{1-xy}\right) = \frac{x+y}{1-xy},$$

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy}.$$

But if two quantities have equal tangents, then they differ from each other by a term divisible by π.

Therefore, indeed,

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy} + \varepsilon\pi. \qquad (*)$$

Let us find out the exact value of ε. Since

$$-\frac{\pi}{2} < \arctan x < +\frac{\pi}{2}, \quad -\frac{\pi}{2} < \arctan y < +\frac{\pi}{2},$$

we have

$$-\pi < \arctan x + \arctan y < +\pi$$

and, consequently,

$$\left| \arctan \frac{x+y}{1-xy} + \varepsilon\pi \right| < \pi.$$

And since

$$-\frac{\pi}{2} < \arctan \frac{x+y}{1-xy} < +\frac{\pi}{2},$$

then $|\varepsilon| < 2$ and, consequently, ε may attain only one of the following three values

$$0, \ +1, \ -1.$$

To find the value of ε let us write the following equality

$$\cos(\arctan x + \arctan y) = \cos\left(\arctan \frac{x+y}{1-xy} + \varepsilon\pi\right).$$

Hence

$$\cos(\arctan x)\cos(\arctan y) - \sin(\arctan x)\sin(\arctan y) = \\ = \cos\left(\arctan \frac{x+y}{1-xy}\right)\cos\varepsilon\pi.$$

On the basis of the results of Problem 23 we have

$$\frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+y^2}} - \frac{x}{\sqrt{1+x^2}} \cdot \frac{y}{\sqrt{1+y^2}} = \\ = \frac{1}{\sqrt{1+\left(\frac{x+y}{1-xy}\right)^2}} \cdot \cos\varepsilon\pi.$$

Consequently

$$\cos\varepsilon\pi = \frac{1-xy}{\sqrt{(1+x^2)(1+y^2)}} \sqrt{1+\left(\frac{x+y}{1-xy}\right)^2}.$$

We have

$$\sqrt{1+\left(\frac{x+y}{1-xy}\right)^2}=\sqrt{\frac{(1+x^2)(1+y^2)}{(1-xy)^2}}=\frac{\sqrt{(1+x^2)(1+y^2)}}{\sqrt{(1-xy)^2}}.$$

But

$$\sqrt{(1-xy)^2}=1-xy \text{ if } 1-xy>0, \text{ i.e. if } xy<1,$$

and

$$\sqrt{(1-xy)^2}=-(1-xy) \text{ if } 1-xy<0, \text{ i.e. if } xy>1.$$

Therefore, $\cos \varepsilon\pi = 1$ if $xy < 1$, and $\cos \varepsilon\pi = -1$ if $xy > 1$. Since $\varepsilon\pi$ can attain only the values 0, π and $-\pi$, it follows that if $xy < 1$, then $\varepsilon = 0$, and if $xy > 1$, then $\varepsilon = \pm 1$. What sign is to be taken is decided in the following way: if $xy > 1$ and $x > 0$, then also $y > 0$, then $\arctan x > 0$ and $\arctan y > 0$, and $\arctan \frac{x+y}{1-xy} < 0$.

The left member of the equality (*) is a positive quantity, consequently, the right member must also be positive, and therefore $\varepsilon\pi$ must exceed zero, and $\varepsilon = +1$. Quite in the same way we make sure that if $xy > 1$ and $x < 0$, $y < 0$, then $\varepsilon = -1$.

26. We have

$$4\arctan\frac{1}{5}=2\arctan\frac{1}{5}+2\arctan\frac{1}{5}=2\arctan\frac{\frac{2}{5}}{1-\frac{1}{25}}=$$

$$=2\arctan\frac{5}{12}=\text{acrtan}\,\frac{5}{12}+\arctan\frac{5}{12}=$$

$$=\arctan\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{25}{144}}=\arctan\frac{120}{119}.$$

Further

$$\arctan\frac{120}{119}+\arctan\left(-\frac{1}{239}\right)=$$

$$=\arctan\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=\arctan 1=\frac{\pi}{4}.$$

27. Using the formula of Problem 25, we easily obtain the result.

28. First of all let us notice, that since $\arcsin x$ is contained between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, and $2\arctan x$ lies between $-\pi$ and $+\pi$, we have

$$-\frac{3\pi}{2} \leqslant 2\arctan x + \arcsin\frac{2x}{1+x^2} \leqslant +\frac{3\pi}{2}.$$

Let us now compute the sine of the required arc, i.e. find what the expression

$$\sin\left(2\arctan x + \arcsin\frac{2x}{1+x^2}\right)$$

is equal to.

We have

$$\sin\left(2\arctan x + \arcsin\frac{2x}{1+x^2}\right) =$$
$$= \sin(2\arctan x)\cos\left(\arcsin\frac{2x}{1+x^2}\right) +$$
$$+ \cos(2\arctan x)\sin\left(\arcsin\frac{2x}{1+x^2}\right).$$

First compute $\sin(2\arctan x)$. Put

$$\arctan x = y, \quad \tan y = x.$$

Then

$$\sin(2\arctan x) = \sin 2y = \tan 2y \cdot \cos 2y.$$

But

$$\tan 2y = \frac{2\tan y}{1-\tan^2 y}, \quad \cos 2y = \frac{1-\tan^2 y}{1+\tan^2 y}.$$

Consequently,

$$\sin(2\arctan x) = \frac{2\tan y}{1+\tan^2 y} = \frac{2x}{1+x^2}.$$

Further

$$\cos\left(\arcsin\frac{2x}{1+x^2}\right) = \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} =$$
$$= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} = \frac{x^2-1}{1+x^2},$$

since $x > 1$.

Further, it is obvious that

$$\cos(2\arctan x) = \frac{1-x^2}{1+x^2},$$
$$\sin\left(\arcsin\frac{2x}{1+x^2}\right) = \frac{2x}{1+x^2},$$

therefore

$$\sin\left(2\arctan x+\arcsin\frac{2x}{1+x^2}\right)=$$

$$=\frac{2x}{1+x^2}\cdot\frac{x^2-1}{1+x^2}+\frac{1-x^2}{1+x^2}\cdot\frac{2x}{1+x^2}=0.$$

Thus, the sine of the required arc is equal to zero, consequently, this arc can have one of the infinite number of values:

$$\ldots,\ -3\pi,\ -2\pi,\ -\pi,\ 0,\ +\pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots.$$

But among these values there are only three ($-\pi$, 0 and π) lying in the required interval between $-\frac{3\pi}{2}$ and $+\frac{3\pi}{2}$. On the other hand, $x>1$ and, consequently, $2\arctan x>0$ and $\arcsin\frac{2x}{1+x^2}>0$, and therefore the required sum

$$2\arctan x+\arcsin\frac{2x}{1+x^2}$$

will also be greater than zero and, consequently, can be equal only to π.

29. It is evident that

$$-\pi\leqslant\arctan x+\arctan\frac{1}{x}\leqslant+\pi.$$

Let us form

$$\sin\left(\arctan x+\arctan\frac{1}{x}\right)$$

The required sine turns out to be equal to (see Problem 23)

$$\sin(\arctan x)\cos\left(\arctan\frac{1}{x}\right)+\cos(\arctan x)\sin\left(\arctan\frac{1}{x}\right)=$$

$$=\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+\frac{1}{x^2}}}+\frac{1}{\sqrt{1+x^2}}\cdot\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}}=$$

$$=\frac{x}{\sqrt{1+x^2}}\cdot\frac{\sqrt{x^2}}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\cdot\frac{\sqrt{x^2}}{x\cdot\sqrt{1+x^2}}=$$

$$=\frac{x^2}{1+x^2}+\frac{1}{1+x^2}=1$$

if $x>0$ (since in this case $\sqrt{x^2}=x$). And if $x<0$, then $\sqrt{x^2}=-x$ and we have $\sin\left(\arctan x+\arctan\frac{1}{x}\right)=-1$.

Hence follows that

$$\arctan x+\arctan\frac{1}{x}=\pm\frac{\pi}{2}+2k\pi,$$

where plus is taken when $x>0$, and minus when $x<0$.

But since, on the other hand, it must be

$$-\pi\leqslant\arctan x+\arctan\frac{1}{x}\leqslant+\pi,$$

our problem has been solved.

30. Compute the expression

$$\sin(\arcsin x+\arcsin y).$$

We have

$$\sin(\arcsin x+\arcsin y)=\sin(\arcsin x)\cos(\arcsin y)+$$
$$+\cos(\arcsin x)\sin(\arcsin y)=x\sqrt{1-y^2}+y\sqrt{1-x^2}.$$

Thus, considering the two arcs

$$\arcsin x+\arcsin y$$

and

$$\arcsin\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right),$$

we may assert that their sines are equal to each other.

However, if

$$\sin\alpha=\sin\beta,\quad 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}=0,$$

and, consequently, either $\frac{\alpha-\beta}{2}=k\pi$ or $\frac{\alpha+\beta}{2}=(2k'+1)\frac{\pi}{2}$ (k and k' integers), i.e. either

$$\alpha=\beta+2k\pi$$

or

$$\alpha=-\beta+(2k'+1)\pi.$$

Therefore we may assert that

$$\arcsin x+\arcsin y=\eta\arcsin\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)+\varepsilon\pi,$$

where $\eta = +1$ if ε is even, and $\eta = -1$ if ε is odd. To determine ε more accurately, let us take cosines of both members. We get

$$\cos(\arcsin x + \arcsin y) = \\ = \cos\left[\eta \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) + \varepsilon\pi\right].$$

Hence

$$\sqrt{1-x^2}\cdot\sqrt{1-y^2} - xy = \\ = (-1)^{\varepsilon}\cos\left[\arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\right].$$

Further

$$\sqrt{1-x^2}\cdot\sqrt{1-y^2} - xy = \\ = (-1)^{\varepsilon}\sqrt{1-\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)^2}.$$

The radicand on the right can be transformed as

$$1-\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)^2 = \\ = 1 - x^2(1-y^2) - y^2(1-x^2) - 2xy\sqrt{1-x^2}\cdot\sqrt{1-y^2} = \\ = (1-x^2)(1-y^2) - 2xy\sqrt{1-x^2}\cdot\sqrt{1-y^2} + x^2y^2 = \\ = \left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right)^2.$$

If it turns out that

$$\sqrt{1-x^2}\cdot\sqrt{1-y^2} - xy > 0,$$

then

$$\sqrt{1-\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)^2} = \\ = \sqrt{\left(\sqrt{1-x^2}\cdot\sqrt{1-y^2} - xy\right)^2} = \sqrt{1-x^2}\cdot\sqrt{1-y^2} - xy.$$

Therefore, in this case

$$(-1)^{\varepsilon} = +1,$$

i.e. ε is even.

And if

$$\sqrt{1-x^2}\sqrt{1-y^2} - xy < 0,$$

then

$$(-1)^{\varepsilon} = -1,$$

and, consequently, ε is odd.

Let us now consider the expression

$$1 - x^2 - y^2.$$

We have

$$1-x^2-y^2 = 1-x^2-y^2+x^2y^2-x^2y^2 =$$
$$= (1-x^2)(1-y^2)-x^2y^2 =$$
$$= (\sqrt{1-x^2}\cdot\sqrt{1-y^2}-xy)(\sqrt{1-x^2}\cdot\sqrt{1-y^2}+xy).$$

The quantity $1 - x^2 - y^2$ can be greater (smaller) than or equal to zero. Let us consider all the three cases.

1° Suppose $1 - x^2 - y^2 > 0$, i.e. $x^2 + y^2 < 1$. If the product of two factors is positive, then these factors are either both positive simultaneously, or both negative simultaneously. And so, we have either

$$\sqrt{1-x^2}\cdot\sqrt{1-y^2}-xy>0, \quad \sqrt{1-x^2}\sqrt{1-y^2}+xy>0$$

or

$$\sqrt{1-x^2}\sqrt{1-y^2}-xy<0, \quad \sqrt{1-x^2}\sqrt{1-y^2}+xy<0.$$

But the second case is impossible, since, adding the last two inequalities, we get

$$\sqrt{1-x^2}\sqrt{1-y^2}<0,$$

which is impossible. If, however, the first two inequalities exist, then

$$\sqrt{1-x^2}\sqrt{1-y^2}-xy>0.$$

Consequently, in this case ε is even.

Thus, if $x^2 + y^2 < 1$, then in our formula ε is even.

2° Let now $1 - x^2 - y^2 < 0$ and, consequently, either

$$\sqrt{1-x^2}\sqrt{1-y^2}-xy>0, \quad \sqrt{1-x^2}\sqrt{1-y^2}+xy<0$$

or

$$\sqrt{1-x^2}\sqrt{1-y^2}-xy<0, \quad \sqrt{1-x^2}\sqrt{1-y^2}+xy>0.$$

But from the first two inequalities we easily obtain $xy < 0$ If this inequality is fulfilled, then it will obligatory be

$$\sqrt{1-x^2}\sqrt{1-y^2}-xy>0,$$

and, consequently, ε is even, From the second pair of inequalities we get $xy > 0$, and ε is odd.

3° Finally, suppose $1 - x^2 - y^2 = 0$. Then again two cases are possible: either $xy \leqslant 0$ or $xy > 0$.

In the first case $\sqrt{1 - x^2} \cdot \sqrt{1 - y^2} - xy > 0$, and, hence, ε is even. Likewise, the second case gives an even ε ($\varepsilon = 0$), since there exists the following relation:

$$\arcsin x + \arcsin \sqrt{1 - x^2} = \frac{\pi}{2} \quad (x > 0).$$

Thus, we can judge whether ε is even or odd. Now let us consider the value of ε. We have

$$|\arcsin x + \arcsin y| < \pi.$$

Consequently

$$\left|\eta \arcsin \left(x \sqrt{1-y^2} + y \sqrt{1-x^2}\right) + \varepsilon\pi\right| < \pi.$$

Hence

$$|\varepsilon| < 2.$$

And so, ε may attain only three values: 0, +1, —1. Comparing the results obtained, we may now assert that

if $x^2 + y^2 \leqslant 1$ or if $xy < 0$, then $\varepsilon = 0$, $\eta = +1$,

and if $x^2 + y^2 > 1$ or if $xy > 0$, then $\varepsilon = \pm 1$, $\eta = -1$. To find out when $\varepsilon = +1$ and when $\varepsilon = -1$, let us notice that at $x > 0$, $y > 0$ $\arcsin x + \arcsin y > 0$ and, consequently,

$$-\arcsin \left(x \sqrt{1-y^2} + y \sqrt{1-x^2}\right) + \varepsilon\pi > 0,$$

and therefore in this case $\varepsilon = +1$. If, however, $x < 0$, $y < 0$, then it is obvious that $\varepsilon = -1$.

31. We have (see Problem 24)

$$\arccos x + \arccos \left(\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right) =$$

$$= \pi - \arcsin x - \arcsin \left(\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right);$$

on the other hand (Problem 30),

$$\arcsin x + \arcsin \left(\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right) = \eta \arcsin \xi + \varepsilon\pi,$$

where

$$\xi = x\sqrt{1-\left(\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)^2}+$$

$$+\left(\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)\sqrt{1-x^2}.$$

But

$$1-\left(\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)^2=\frac{1}{4}(\sqrt{1-x^2}-\sqrt{3}\,x)^2,$$

and since $x \geqslant \frac{1}{2}$, we have $4x^2 \geqslant 1$: $3x^2 \geqslant 1-x^2$ and $\sqrt{3}\,x \geqslant \sqrt{1-x^2}$.

Therefore

$$\sqrt{1-\left(\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)^2}=\frac{1}{2}\sqrt{(\sqrt{1-x^2}-\sqrt{3}\,x)^2}=$$

$$=\frac{1}{2}(\sqrt{3}\,x-\sqrt{1-x^2})$$

and $\xi = \frac{\sqrt{3}}{2}$.

Consequently

$$\arcsin \xi = \frac{\pi}{3}.$$

The only thing which is left is to find η and ε (see Problem 30).

Let us prove that

$$x^2+\left(\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)^2 > 1.$$

We have

$$x^2+\frac{x^2}{4}+\frac{3}{4}(1-x^2)+\frac{1}{2}\sqrt{3}\,x\sqrt{1-x^2} \geqslant$$

$$\geqslant \frac{3}{4}+\frac{1}{2}x^2+\frac{1}{2}(1-x^2)=\frac{5}{4}.$$

Consequently,

$$\eta = -1, \quad \varepsilon = +1.$$

Therefore,

$$\arccos x + \arccos\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)=\pi-\left(-\frac{\pi}{3}+\pi\right)=\frac{\pi}{3}.$$

32. We have $\tan A = \frac{1}{7}$, $\tan B = \frac{1}{3}$. Let us compute $\cos 2A$. Since

$$1 + \tan^2 A = \frac{1}{\cos^2 A},$$

we have

$$\frac{1}{\cos^2 A} = 1 + \frac{1}{49} = \frac{50}{49} \text{ and } \cos^2 A = \frac{49}{50}.$$

But

$$\cos 2A = 2\cos^2 A - 1 = \frac{98}{50} - 1 = \frac{24}{25}.$$

Further

$$\sin 4B = 2 \sin 2B \cos 2B.$$

But

$$\cos 2B = 2\cos^2 B - 1 = \frac{2}{1+\tan^2 B} - 1 = \frac{4}{5},$$

$$\sin 2B = 2 \sin B \cos B = 2 \tan B \cos^2 B = \frac{2 \tan B}{1 + \tan^2 B} = \frac{3}{5}.$$

Consequently,

$$\sin 4B = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25} \quad \text{and} \quad \sin 4B = \cos 2A.$$

33. By hypothesis we have

$$(a+b)^2 = 9ab \text{ or } \left(\frac{a+b}{3}\right)^2 = ab.$$

The rest is obvious.

34. Put

$$\log_a n = x, \quad \log_{ma} n = y.$$

Then

$$a^x = n, \quad m^y a^y = n.$$

Hence

$$a^x = m^y \cdot a^y, \quad a^{\frac{x}{y}} = ma.$$

Taking logarithms of this last equality to the base a, we get the required result.

35. Put

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z} = \frac{1}{t}.$$

Then

$$\log x = tx\,(y + z - x),\ \log y = ty\,(z + x - y),$$
$$\log z = tz\,(x + y - z).$$

Hence

$$y \log x + x \log y = 2txyz,\ y \log z + z \log y = 2txyz,$$
$$z \log x + x \log z = 2txyz.$$

Consequently

$$y \log x + x \log y = y \log z + z \log y = z \log x + x \log z,$$
$$\log x^y y^x = \log z^y y^z = \log x^z z^x.$$

Finally

$$x^y y^x = z^y y^z = x^z z^x.$$

36. 1° Put $\log_b a = x$. Then

$$b^x = a.$$

Taking logarithms of this equality to the base a, we get

$$x \log_a b = 1.$$

But $x = \log_b a$. Consequently, indeed, $\log_b a \log_a b = 1$.
2° We have

$$a^{\log_a b} = b.$$

Therefore

$$a^{\frac{\log_b (\log_b a)}{\log_b a}} = \left(a^{\frac{1}{\log_b a}}\right)^{\log_b(\log_b a)} = \left(a^{\log_a b}\right)^{\log_b (\log_b a)} =$$
$$= b^{\log_b (\log_b a)} = \log_b a.$$

37. From the given relations it follows that

$$y^{1-\log x} = 10,\quad z^{1-\log y} = 10.$$

Taking logarithms of these equalities to the base 10, we get

$$(1 - \log x) \log y = 1,\quad (1 - \log y) \log z = 1.$$

whence

$$\log x = 1 - \frac{1}{\log y} = 1 - \frac{1}{1 - \frac{1}{\log z}} = \frac{1}{1 - \log z}$$

and, consequently,

$$x = 10^{\frac{1}{1-\log z}}.$$

38. The original equality yields

$$a^2 = (c - b)(c + b).$$

Hence

$$2 \log_{c+b} a = \log_{c+b}(c - b) + 1,$$
$$2 \log_{c-b} a = \log_{c-b}(c + b) + 1.$$

Multiplying these equalities, we find

$$4 \log_{c+b} a \cdot \log_{c-b} a = \log_{c+b}(c - b) + \log_{c-b}(c + b) + 1 + \log_{c+b}(c - b) \log_{c-b}(c + b).$$

However,

$$\log_{c-b}(c + b) \log_{c+b}(c - b) = 1.$$

Therefore

$$4 \log_{c+b} a \log_{c-b} a = 2 \log_{c+b} a - 1 + 2 \log_{c-b} a - 1 + 2.$$

Finally

$$\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a.$$

39. Put

$$\log_a N = x, \quad \log_c N = y, \quad \log_{\sqrt{ac}} N = z.$$

The last equality yields

$$(ac)^{\frac{z}{2}} = N.$$

Hence

$$\log_a N = \frac{z}{2}(1 + \log_a c), \quad \log_c N = \frac{z}{2}(1 + \log_c a).$$

Therefore

$$\frac{2x}{z} - 1 = \log_a c, \quad \frac{2y}{z} - 1 = \log_c a.$$

Consequently

$$\left(\frac{2x}{z} - 1\right)\left(\frac{2y}{z} - 1\right) = 1$$

or

$$\frac{x}{y} = \frac{x - z}{z - y}.$$

40. We have

$$\log_{a_1 a_2 \ldots a_n} x = \frac{1}{\log_x a_1 a_2 \ldots a_n} = \frac{1}{\log_x a_1 + \log_x a_2 + \ldots + \log_x a_n} = $$
$$= \frac{1}{\frac{1}{\log_{a_1} x} + \frac{1}{\log_{a_2} x} + \ldots + \frac{1}{\log_{a_n} x}}.$$

41. Let

$$a_n = aq^n, \quad b_n = b + nd.$$

Then

$$\log a_n = \log a + n \log q, \ \log a_n - b_n =$$
$$= \log a + n \log q - b - nd = \log a - b.$$

Hence

$$n \log q - nd = 0, \quad \log_\beta q = d, \quad \beta^d = q.$$

And so

$$\beta = q^{\frac{1}{d}}.$$

SOLUTIONS TO SECTION 4

1. We have

$$\left(\frac{x-ab}{a+b} - c\right) + \left(\frac{x-ac}{a+c} - b\right) + \left(\frac{x-bc}{b+c} - a\right) = 0.$$

Hence

$$\frac{x-ab-ac-bc}{a+b} + \frac{x-ac-ab-bc}{a+c} + \frac{x-bc-ab-ac}{b+c} = 0$$

or

$$(x-ab-ac-bc)\left(\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c}\right) = 0.$$

Assuming that

$$\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c}$$

is not equal to zero, we obtain

$$x = ab + ac + bc.$$

If, however,

$$\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c} = 0,$$

then the given equation turns into an identity which holds true for any value of x.

2. Rewrite the equation as follows

$$\left(\frac{x-a}{bc} - \frac{1}{b} - \frac{1}{c}\right) + \left(\frac{x-b}{ac} - \frac{1}{a} - \frac{1}{c}\right) + \left(\frac{x-c}{ab} - \frac{1}{a} - \frac{1}{b}\right) = 0.$$

We have

$$\frac{x-a-b-c}{bc}+\frac{x-b-a-c}{ac}+\frac{x-c-a-b}{ab}=0.$$

Hence

$$(x-a-b-c)\left(\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}\right)=0,$$

and, consequently,

$$x=a+b+c.$$

It is assumed, of course, that none of the quantities a, b and c, as also $\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab}$ is equal to zero.

3. If we put in our equation

$$6x+2a=A,\ 3b+c=B,\ 2x+6a=C,\ b+3c=D,$$

then it is rewritten in the following way

$$\frac{A+B}{A-B}=\frac{C+D}{C-D}.$$

Adding unity to both members of the equation, we find

$$\frac{2A}{A-B}=\frac{2C}{C-D}.$$

Likewise, subtracting unity, we get

$$\frac{2B}{A-B}=\frac{2D}{C-D}.$$

Dividing the last equalities termwise, we have

$$\frac{A}{B}=\frac{C}{D},$$

i.e.

$$\frac{6x+2a}{3b+c}=\frac{2x+6a}{b+3c}.$$

Hence

$$\left(\frac{6}{3b+c}-\frac{2}{b+3c}\right)x=\left(\frac{6}{b+3c}-\frac{2}{3b+c}\right)a.$$

Finally

$$x=\frac{ab}{c}.$$

4. Add 3 to both members of the equation and rewrite it in the following way

$$\left(\frac{a+b-x}{c}+1\right)+\left(\frac{a+c-x}{b}+1\right)+\left(\frac{b+c-x}{a}+1\right)=$$
$$=4-\frac{4x}{a+b+c}.$$

Hence

$$(a+b+c-x)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=4\,\frac{a+b+c-x}{a+b+c}.$$

Consequently

$$(a+b+c-x)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{4}{a+b+c}\right)=0$$

and, finally,

$$x=a+b+c.$$

5. Taking $\sqrt[p]{b+x}$ outside the brackets in the left member, we get

$$\sqrt[p]{b+x}\,\frac{b+x}{bx}=\frac{c}{a}\sqrt[p]{x}.$$

Consequently,

$$\frac{(b+x)^{1+\frac{1}{p}}}{x^{1+\frac{1}{p}}}=\frac{bc}{a}.$$

Hence

$$\left(\frac{b+x}{x}\right)^{\frac{p+1}{p}}=\frac{bc}{a},\qquad \frac{b+x}{x}=\left(\frac{bc}{a}\right)^{\frac{p}{p+1}}$$

Further

$$\frac{b}{x}=\left(\frac{bc}{a}\right)^{\frac{p}{p+1}}-1,\qquad x=\frac{b}{\left(\frac{bc}{a}\right)^{\frac{p}{p+1}}-1}.$$

6. 1° Squaring both members of the given equation, we find

$$x+1+x-1+2\sqrt{x^2-1}=1.$$

Consequently,

$$2\sqrt{x^2-1}=1-2x,$$
$$4x^2-4=1+4x^2-4x,$$
$$x=\frac{5}{4}.$$

Since squaring leads, generally speaking, to an equation not equivalent to the given one, or rather to such an equation which in addition to the roots of the given equation may have other roots different from them (so-called extraneous roots), it is necessary to check, by substitution, whether $\frac{5}{4}$ is really the root of the original equation. The check shows that $\frac{5}{4}$ does not satisfy the original equation (here, as before, we consider only principal values of the roots).

2° Carrying out all necessary transformations similar to the previous ones, we find that $x=\frac{5}{4}$ is the root of our equation.

7. Cube both members of the given equation, taking the formula for the cube of a sum in the following form

$$(A+B)^3=A^3+B^3+3AB\,(A+B).$$

We have

$$a+\sqrt{x}+a-\sqrt{x}+3\sqrt[3]{a^2-x}\,(\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}})=b.$$

Since

$$\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}}=\sqrt[3]{b},$$

we have

$$2a+3\sqrt[3]{a^2-x}\cdot\sqrt[3]{b}=b,\qquad x=a^2-\frac{(b-2a)^3}{27b}.$$

We assume that a and b are such that

$$a^2-\frac{(b-2a)^3}{27b}\geqslant 0.$$

Since the equality of cubes of two real numbers also means the equality of the numbers themselves, the found value of x satisfies the original equation as well.

8. Squaring both members of the equation, we find

$$-\sqrt{x^4-x^2}=x^2-2x.$$

Hence

$$x^4-x^2-x^2(x-2)^2=0,$$

$$x^2[x^2-1-x^2-4+4x]=x^2(4x-5)=0.$$

Thus, the last equation has two roots $x=0$ and $x=\frac{5}{4}$. Substituting them into the original equation, we see that the unique root of this equation is

$$x=\frac{5}{4}.$$

9. Getting rid of the denominator, we obtain

$$(\sqrt{a}+\sqrt{x-b})\sqrt{b}=\sqrt{a}(\sqrt{b}+\sqrt{x-a})$$

or

$$\sqrt{b(x-b)}=\sqrt{a(x-a)}, \qquad b(x-b)=a(x-a), \qquad x=a+b.$$

As is easily seen, this value of x is also the root of the original equation.

10. Multiplying both the numerator and denominator by $\sqrt{a+x}+\sqrt{a-x}$, we get

$$(\sqrt{a+x}+\sqrt{a-x})^2=2x\sqrt{b}.$$

Hence

$$\sqrt{a^2-x^2}=x\sqrt{b}-a.$$

Squaring both members of this equality, we find two roots

$$x=0, \quad x=\frac{2a\sqrt{b}}{1+b}.$$

However, the first of these values is not the root of the original equation, the second one will be its root if

$$b\geqslant 1.$$

Indeed, we have

$$\sqrt{a+x}=\sqrt{a+\frac{2a\sqrt{b}}{1+b}}=\sqrt{a}\sqrt{\frac{(1+\sqrt{b})^2}{1+b}}=\sqrt{a}\,\frac{1+\sqrt{b}}{\sqrt{1+b}},$$

$$\sqrt{a-x}=\sqrt{a-\frac{2a\sqrt{b}}{1+b}}=\sqrt{a}\sqrt{\frac{(\sqrt{b}-1)^2}{1+b}}=$$

$$=\sqrt{a}\,\frac{\sqrt{b}-1}{\sqrt{1+b}}\quad (\text{if } \sqrt{b}-1\geqslant 0).$$

Substituting the obtained values for $\sqrt{a+x}$ and $\sqrt{a-x}$ into the original equation, we make sure that our assertion is true.

11. Adding all the given equations, we have

$$x+y+z+v=\frac{a+b+c+d}{3}.$$

Consequently

$$v=(x+y+z+v)-(x+y+z)=\frac{a+b+c+d}{3}-a=$$

$$=\frac{b+c+d-2a}{3}.$$

Likewise, we obtain

$$z=\frac{a+c+d-2b}{3},\quad y=\frac{a+b+d-2c}{3},\quad x=\frac{a+b+c-2d}{3}.$$

12. Adding all the four equations, we get

$$4x_1=2a_1+2a_2+2a_3+2a_4,$$

$$x_1=\frac{a_1+a_2+a_3+a_4}{2}.$$

Multiplying the last two equations by -1, and then adding all the four equations, we find

$$x_2=\frac{a_1+a_2-a_3-a_4}{2}.$$

Similarly, we get

$$x_3=\frac{a_1-a_2+a_3-a_4}{2},\quad x_4=\frac{a_1-a_2-a_3+a_4}{2}.$$

13. Put $x+y+z+v=s$. Then the system is rewritten as follows

$$ax+m(s-x)=k$$
$$by+m(s-y)=l$$
$$cz+m(s-z)=p$$
$$dv+m(s-v)=q$$

so that

$$ms+x(a-m)=k,\quad ms+y(b-m)=l,$$
$$ms+z(c-m)=p,\quad ms+v(d-m)=q.$$

Hence

$$x=\frac{k}{a-m}-\frac{m}{a-m}s,\quad y=\frac{l}{b-m}-\frac{m}{b-m}s,\quad z=\frac{p}{c-m}-\frac{m}{c-m}s,$$
$$v=\frac{q}{d-m}-\frac{m}{d-m}s. \qquad (*)$$

Adding these equalities termwise, we find

$$s=\frac{k}{a-m}+\frac{l}{b-m}+\frac{p}{c-m}+\frac{q}{d-m}-$$
$$-ms\left(\frac{1}{a-m}+\frac{1}{b-m}+\frac{1}{c-m}+\frac{1}{d-m}\right).$$

Consequently

$$s\left[1+m\left(\frac{1}{a-m}+\frac{1}{b-m}+\frac{1}{c-m}+\frac{1}{d-m}\right)\right]=$$
$$=\frac{k}{a-m}+\frac{l}{b-m}+\frac{p}{c-m}+\frac{q}{d-m}.$$

Wherefrom we find s, and then from the equalities (*) we obtain the required values of the unknowns x, y, z and v.

14. Put

$$\frac{x_1-a_1}{m_1}=\frac{x_2-a_2}{m_2}=\ldots=\frac{x_p-a_p}{m_p}=\lambda.$$

Hence

$$x_1=a_1+m_1\lambda,$$
$$x_2=a_2+m_2\lambda,$$
$$\ldots\ldots\ldots\ldots$$
$$x_p=a_p+m_p\lambda.$$

Substituting these into the last one of the given equations, we get

$$x_1 + x_2 + \ldots + x_p = a =$$
$$= (a_1 + a_2 + \ldots + a_p) + \lambda (m_1 + m_2 + \ldots + m_p).$$

Consequently,

$$\lambda = \frac{a - a_1 - a_2 - \ldots - a_p}{m_1 + m_2 + \ldots + m_p},$$

and then we readily get the values of

$$x_1, x_2, \ldots, x_p.$$

15. If we put

$$\frac{1}{x} = x', \quad \frac{1}{y} = y', \quad \frac{1}{z} = z', \quad \frac{1}{v} = v',$$

then the solution of this system is reduced to that of Problem 11. Using the result of Problem 11, we easily obtain

$$x = \frac{3}{a+b+c-2d}, \quad y = \frac{2}{a+b+d-2c},$$
$$z = \frac{3}{a+c+d-2b}, \quad v = \frac{3}{b+c+d-2a}.$$

16. Dividing the first equation by ab, the second by ac and the third by bc (assuming $abc \neq 0$), we get

$$\frac{y}{b} + \frac{x}{a} = \frac{c}{ab}, \quad \frac{x}{a} + \frac{z}{c} = \frac{b}{ac}, \quad \frac{z}{c} + \frac{y}{b} = \frac{a}{bc}.$$

Adding all these equations termwise, we find

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{2}\left(\frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}\right).$$

Hence

$$\frac{z}{c} = \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) - \left(\frac{x}{a} + \frac{y}{b}\right) = \frac{1}{2}\left(\frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}\right) - \frac{c}{ab}.$$

Consequently, $\frac{z}{c} = \frac{a^2 + b^2 - c^2}{2abc}$, i.e. $z = \frac{a^2 + b^2 - c^2}{2ab}$ and then analogously

$$y = \frac{a^2 + c^2 - b^2}{2ac}, \quad x = \frac{b^2 + c^2 - a^2}{2bc}.$$

17. First of all we have an obvious solution $x = y = z = 0$. Let us now look for nonzero solutions, i.e. for such in which x, y, z are not equal to zero. Dividing the first of the given equations by yz, the second by zx and the third by xy, we obtain

$$\frac{c}{z}+\frac{b}{y}=2d, \quad \frac{a}{x}+\frac{c}{z}=2d', \quad \frac{b}{y}+\frac{a}{x}=2d''.$$

Hence

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=d+d'+d''.$$

Therefore

$$\frac{a}{x}=d'+d''-d, \quad \frac{b}{y}=d+d''-d', \quad \frac{c}{z}=d+d'-d''.$$

Finally

$$x=\frac{a}{d'+d''-d}, \quad y=\frac{b}{d+d''-d'}, \quad z=\frac{c}{d+d'-d''}.$$

18. Rewrite the system in the following way

$$\frac{ay+bx}{xy}=\frac{1}{c}, \quad \frac{az+cx}{xz}=\frac{1}{b}, \quad \frac{bz+cy}{yz}=\frac{1}{a}.$$

Hence

$$\frac{a}{x}+\frac{b}{y}=\frac{1}{c}, \quad \frac{a}{x}+\frac{c}{z}=\frac{1}{b}, \quad \frac{b}{y}+\frac{c}{z}=\frac{1}{a}.$$

Consequently (see the preceding problem)

$$x=\frac{2a^2bc}{ac+ab-bc}, \quad y=\frac{2ab^2c}{bc+ab-ac}, \quad z=\frac{2abc^2}{bc+ac-ab}.$$

19. The obvious solution is $x=y=z=0$. Dividing both members of each equation of our system by xyz, we get

$$\frac{1}{xz}+\frac{1}{xy}-\frac{1}{yz}=\frac{1}{a^2}, \quad \frac{1}{xy}+\frac{1}{yz}-\frac{1}{xz}=\frac{1}{b^2},$$
$$\frac{1}{yz}+\frac{1}{xz}-\frac{1}{xy}=\frac{1}{c^2}.$$

Adding pairwise, we find

$$\frac{2}{xy}=\frac{1}{a^2}+\frac{1}{b^2}, \quad \frac{2}{yz}=\frac{1}{b^2}+\frac{1}{c^2}, \quad \frac{2}{xz}=\frac{1}{a^2}+\frac{1}{c^2}.$$

Consequently

$$xy=\frac{2a^2b^2}{a^2+b^2}, \quad yz=\frac{2b^2c^2}{b^2+c^2}, \quad xz=\frac{2a^2c^2}{a^2+c^2}. \qquad (*)$$

Multiplying the equalities, we obtain

$$x^2y^2z^2 = \frac{8a^4b^4c^4}{(a^2+b^2)(b^2+c^2)(a^2+c^2)}.$$

Hence

$$xyz = \pm \frac{2\sqrt{2}\, a^2b^2c^2}{\sqrt{(a^2+b^2)(b^2+c^2)(a^2+c^2)}}.$$

Using the equality

$$xy = \frac{2a^2b^2}{a^2+b^2},$$

we find for z two values which differ in the sign. By the obtained value of z we find the corresponding values of y and x from the equalities (*). Thus, we get two sets of values for x, y and z satisfying our equation.

20. Adding all the three equations, we find

$$(x+y+z)(a+b+c) = 0.$$

Hence

$$x+y+z=0,$$

whence

$$x = \frac{a-b}{a+b+c}, \quad y = \frac{a-c}{a+b+c}, \quad z = \frac{b-a}{a+b+c}.$$

21. Adding all the three equations termwise, we get

$$(b+c)x + (c+a)y + (a+b)z = 2a^3 + 2b^3 + 2c^3.$$

Using the given equations in succession, we find

$$2(b+c)x = 2b^3 + 2c^3, \quad 2(c+a)y = 2a^3 + 2c^3,$$
$$2(a+b)z = 2a^3 + 2b^3,$$

whence

$$x = b^2 - bc + c^2, \quad y = a^2 - ac + c^2, \quad z = a^2 - ab + b^2.$$

22. Consider the following equality

$$\frac{x}{a+\theta} + \frac{y}{b+\theta} + \frac{z}{b+\theta} - 1 = -\frac{(\theta-\lambda)(\theta-\mu)(\theta-\nu)}{(\theta+a)(\theta+b)(\theta+c)}.$$

Let us transform the equality, by reducing its terms to a common denominator and then rejecting the latter. We get a second-degree polynomial in θ with coefficients depending on x, y, z, λ, μ, ν, a, b, c, which is equal to zero. If now we substitute successivly λ, μ and ν for θ into the original expression, then, by virtue of the given equations, this expression (and, consequently, the second-degree polynomial) vanishes. However, if a *second-degree* polynomial becomes zero at *three different* values of the variable, then it is identically equal to zero (see Sec. 2) and, consequently, the equality

$$\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}-1=-\frac{(\theta-\lambda)(\theta-\mu)(\theta-\nu)}{(\theta+a)(\theta+b)(\theta+c)}$$

(by virtue of existence of the three given equations) is an identity with respect to θ, i.e. it holds for any values of θ.

Multiplying both members of this equality by $a+\theta$, put $\theta=-a$. Then we find

$$x=\frac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}.$$

Likewise we get

$$y=\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b-c)(b-a)}, \quad z=\frac{(c+\lambda)(c+\mu)(c+\nu)}{(c-a)(c-b)}.$$

Of course, we assume here that the given quantities λ, μ, ν, as also a, b and c, are not equal to one another.

23. The given equations show that the polynomial

$$\alpha^3+x\alpha^2+y\alpha+z$$

vanishes at three different values of a, namely at $\alpha=a$, at $\alpha=b$ and at $\alpha=c$ (assuming that a, b and c are not equal to one another).

Set up a difference

$$\alpha^3+x\alpha^2+y\alpha+z-(\alpha-a)(\alpha-b)(\alpha-c).$$

This difference also becomes zero at α equal to a, b, c. Expanding this expression in powers of α, we obtain

$$(x+a+b+c)\alpha^2+(y-ab-ac-bc)\alpha+$$
$$+z+abc.$$

This *second-degree* trinomial in α vanishes at *three different* values of α, and therefore it equals zero identically and, consequently, all its coefficients are equal to zero, i.e.

$$x+a+b+c=0, \quad y-ab-ac-bc=0,$$
$$z+abc=0.$$

Hence

$$x = -(a + b + c),$$
$$y = ab + ac + bc,$$
$$z = -abc$$

is the solution of our system.

24. We find similarly

$$t = -(a + b + c + d),$$
$$x = ab + ac + ad + bc + bd + cd,$$
$$y = -(abc + abd + acd + bcd),$$
$$z = abcd.$$

25. Multiplying the first equation by r, the second by p, the third by q and the fourth by 1 and adding, we get

$$(a^3 + a^2q + ap + r)\,x + (b^3 + b^2q + bp + r)\,y + + (c^3 + c^2q + cp + r)\,z + (d^3 + d^2q + dp + r)\,u = = mr + np + kq + l.$$

Let us choose the quantities r, p and q so that the following equalities take place

$$b^3 + b^2q + bp + r = 0,$$
$$c^3 + c^2q + cp + r = 0,$$
$$d^3 + d^2q + dp + r = 0.$$

Hence, we obtain (see Problem 23)

$$q = -(b + c + d), \quad p = bc + bd + cd, \quad r = -bcd,$$

and, consequently

$$x = \frac{N}{a^3 + a^2q + ap + r} = \frac{N}{(a-b)(a-c)(a-d)},$$

where

$$N = -mbcd + n(bc + bd + cd) - k(b + c + d) + l.$$

As to the equality

$$a^3 + a^2q + ap + r = (a - b)(a - c)(a - d),$$

it follows readily from the identity

$$\alpha^3 + q\alpha^2 + p\alpha + r = (\alpha - b)(\alpha - c)(\alpha - d).$$

To find the variable y, the quantities q, p and r are so chosen that the following equalities take place

$$a^3 + a^2q + ap + r = 0,$$
$$c^3 + c^2q + cp + r = 0,$$
$$d^3 + d^2q + dp + r = 0.$$

The remaining variables are found analogously.

26. Put

$$x_1 + x_2 + \ldots + x_n = s.$$

Adding the equations term by term, we get

$$s + 2s + 3s + \ldots + ns = a_1 + a_2 + \ldots + a_n.$$

But

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$ (an arithmetic progression).

Therefore

$$s = \frac{2}{n(n+1)} (a_1 + a_2 + \ldots + a_n) = A \text{ (for brevity).}$$

Subtracting now the second equation from the first one, we find

$$x_1 + x_2 + x_3 + \ldots + x_n - nx_1 = a_1 - a_2.$$

Hence

$$nx_1 = A + a_2 - a_1$$

and

$$x_1 = \frac{A + a_2 - a_1}{n}.$$

Subtracting the third equation from the second, we get

$$x_2 = \frac{A + a_3 - a_2}{n}$$

and so on.

27. Put

$$x_1 + x_2 + \ldots + x_n = s.$$

Then we have

$$-s + 2x_1 = 2a, \qquad -s + 4x_2 = 4a,$$
$$-s + 8x_3 = 8a, \ldots, \qquad -s + 2^n x_n = 2^n a.$$

Hence

$$x_1 = a + \frac{s}{2}, \; x_2 = a + \frac{s}{4}, \; x_3 = a + \frac{s}{8}, \; \ldots, \; x_n = a + \frac{s}{2^n}.$$

Adding these equalities, we get

$$s = na + s\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right).$$

But

$$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Therefore

$$s = 2^n na.$$

Consequently

$$x_1 = a + \frac{s}{2} = a + 2^{n-1}na = a(1 + n \cdot 2^{n-1}),$$

$$x_2 = a + \frac{s}{4} = a + 2^{n-2}na = a(1 + n \cdot 2^{n-2}) \quad \text{and so on.}$$

28. Let

$$x_1 + x_2 + x_3 + \ldots + x_n = s = 1.$$

Then

$$s - x_2 = 2, \quad s - x_3 = 3, \; \ldots, \; s - x_{n-1} = n - 1,$$
$$s - x_n = n.$$

Consequently (since $s = 1$)

$$x_2 = -1, \quad x_3 = -2, \; \ldots, \; x_n = -(n-1).$$

Hence

$$x_2 + x_3 + \ldots + x_n = -[(1 + 2 + \ldots + (n-1)] = $$
$$= -\frac{n(n-1)}{2}.$$

Finally

$$x_1 = 1 - (x_2 + x_3 + \ldots + x_n) = 1 + \frac{n(n-1)}{2}.$$

29. Suppose the equations are compatible, i.e. there exists such a value of x at which both equations are satisfied. Substituting this value of x into the given equations, we get the following identities

$$ax + b = 0, \quad a'x + b' = 0.$$

Multiply the first of them by b', and the second by b. Subtracting termwise the obtained equalities, we find

$$(ab' - a'b)\,x = 0.$$

If the common solution for x is nonzero, then it actually follows from the last equality

$$ab' - a'b = 0.$$

If the common solution is equal to zero, then from the original equation it follows that

$$b = b' = 0,$$

and therefore in this case also

$$ab' - a'b = 0.$$

And so, in both cases, if the two given equations have a common solution, then

$$ab' - a'b = 0.$$

Hence, conversely if the condition

$$ab' - a'b = 0$$

is satisfied, the two given equations have a common root (the coefficients of the equations are proportional), and, consequently, they are compatible.

30. To prove that the given systems are equivalent it is necessary to prove that each solution of one of the systems is simultaneously a solution for the other system. Indeed, it is apparent, that each solution of the first system is at the same time a solution for the second system. It only remains to prove that each solution of the second system will also be a solution for the first system. Suppose a pair of numbers x and y is the solution of the second system, i.e. we have identically

$$l\xi + l'\xi' = 0,$$
$$m\xi + m'\xi' = 0,$$

where

$$\xi = ax + by + c, \ \xi' = a'x + b'y + c'.$$

Multiplying the first equality by m' and the second by l', and subtracting them termwise, we find

$$(lm' - ml')\,\xi = 0.$$

Likewise, multiplying the first equality by m and the second by l, and subtracting, we get

$$(lm' - ml')\,\xi' = 0.$$

But since, by hypothesis,

$$lm' - ml' \neq 0,$$

it follows from the last two equalities that

$$\xi = 0$$

and

$$\xi' = 0,$$

i.e.

$$ax + by + c = 0$$

and

$$a'x + b'y + c' = 0.$$

Thus, the pair of numbers x and y, which is the solution of the second system, is simultaneously the solution of the first system.

31. Multiplying the first equation by b' and the second by b, and subtracting termwise, we find

$$(ab' - a'b)\,x + cb' - c'b = 0.$$

We get similarly

$$(ab' - a'b)\,y + c'a - a'c = 0.$$

These two equations are equivalent to the original ones. It is evident that if $ab' - a'b \neq 0$, then there exists one and only one pair of values of x and y satisfying the last two equalities, and, consequently, the original system as well.

32. Multiplying the first equality by b' and the second by b, and subtracting, we find

$$(ab' - a'b)\,x = 0.$$

Since, by hypothesis, $ab' - a'b \neq 0$, it follows that $x = 0$. In the same way we prove that $y = 0$.

33. From the first two equations we get

$$x = \frac{c'b - cb'}{ab' - a'b}, \quad y = \frac{a'c - c'a}{ab' - a'b}.$$

If the three equations are compatible, then a pair of numbers x and y being the solution of the system of the first two equations must also satisfy the third equation. Therefore, if the three given equations are compatible, then there exists the following relation

$$a'' \frac{c'b - cb'}{ab' - a'b} + b'' \frac{a'c - c'a}{ab' - a'b} + c'' = 0$$

or

$$a''(c'b - cb') + b''(a'c - c'a) + c''(ab' - a'b) = 0. \qquad (*)$$

Conversely, the existence of this relation means that a solution, which satisfies the first two equations, satisfies the third one as well. This relation may be rewritten in the following ways

$$a'(cb'' - c''b) + b'(ac'' - ca'') + c'(ba'' - b''a) = 0,$$

$$a(c''b' - c'b'') + b(a''c' - c''a') + c(b''a' - a''b') = 0.$$

Hence it follows that the solution of each pair of the three equations is necessarily the solution of the third equation, i.e. our system is compatible provided the condition (*) is observed.

34. Subtracting from the first equality the second, and then the third one, we find

$$(a - b)y + (a^2 - b^2)z = 0, \quad (a - c)y + (a^2 - c^2)z = 0.$$

Since $a - b \neq 0$ and $a - c \neq 0$, we have the following equalities

$$y + (a + b)z = 0, \quad y + (a + c)z = 0.$$

Subtracting them term by term, we have

$$(b - c)z = 0.$$

But by hypothesis $b - c \neq 0$, therefore $z = 0$. Substituting this value into one of the last two equations, we find $y = 0$. Finally, making use of one of the original equations, we get

$$x = 0.$$

35. Multiplying the first equality by B_1 and the second one by B, and subtracting them termwise, we get

$$(AB_1 - A_1B)x + (CB_1 - C_1B)z = 0. \qquad (1)$$

We find analogously

$$(AC_1 - A_1C)x + (BC_1 - B_1C)y = 0. \qquad (2)$$

Suppose none of the expressions

$$AB_1 - A_1B, \quad CB_1 - C_1B, \quad AC_1 - A_1C$$

is equal to zero. Then we get

$$\frac{x}{C_1B - CB_1} = \frac{z}{AB_1 - A_1B}.$$

[multiplying both members of the first equality by the product

$$(AB_1 - A_1B)(C_1B - CB_1)]$$

and

$$\frac{x}{C_1B - CB_1} = \frac{y}{CA_1 - AC_1}.$$

Thus, in this case the required proportion really takes place.

Let now *one and only one* of the expressions

$$AB_1 - A_1B, \quad CB_1 - C_1B, \quad AC_1 - A_1C$$

vanish. Put, for instance, $CB_1 - C_1B = 0$. Then from equalities (1) and (2) we get $x = 0$. Further, suppose that two of the mentioned expressions, for instance, $C_1B - CB_1$ and $CA_1 - AC_1$ are equal to zero, and the third one, i.e. $AB_1 - A_1B$ is nonzero. We then find $x = y = 0$. In these cases our proportion, or, more precisely, three equalities,

$$x = \lambda (C_1B - CB_1),$$
$$y = \lambda (CA_1 - AC_1),$$
$$z = \lambda (AB_1 - A_1B),$$

will also take place.

Thus, in these cases two given equations determine the variables x, y and z "accurate to the common factor of proportionality".

If all the three quantities

$$AB_1 - A_1B, \quad CB_1 - C_1B \quad \text{and} \quad AC_1 - A_1C$$

are equal to zero, then there exists the following proportion

$$\frac{A}{A_1} = \frac{B}{B_1} = \frac{C}{C_1}.$$

In this case the two equations (forming a system) turn into one, and nothing definite can be said about the values of the variables x, y and z which satisfy this equation.

36. From the first two equations (see the preceding problem) we get

$$\frac{x}{ac-b^2}=\frac{y}{bc-a^2}=\frac{z}{ab-c^2}.$$

Hence

$$x=\lambda(ac-b^2),\quad y=\lambda(bc-a^2),\quad z=\lambda(ab-c^2).$$

Substituting these values into the third equation, we find

$$b(ac-b^2)+a(bc-a^2)+c(ab-c^2)=0$$

or

$$a^3+b^3+c^3-3abc=0.$$

37. Multiplying the first two equations, we get

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}.$$

The same result is obtained by multiplying the third equation by the fourth one, which shows that if there exist any three of the given equations, then there also exists a fourth one, i.e. the system is compatible.

To determine the values of x, y and z satisfying the given system proceed in the following way: equating the right members of the first and the third equations, find

$$\lambda\left(1+\frac{y}{b}\right)=\mu\left(1-\frac{y}{b}\right).$$

Solving this equation with respect to y, we have

$$y=b\frac{\mu-\lambda}{\mu+\lambda}.$$

Substituting this into the first two equations, we get

$$\frac{x}{a}+\frac{z}{c}=\frac{2\lambda\mu}{\mu+\lambda},\quad \frac{x}{a}-\frac{z}{c}=\frac{2}{\mu+\lambda}.$$

Hence

$$x=a\frac{\lambda\mu+1}{\mu+\lambda},\quad z=c\frac{\lambda\mu-1}{\mu+\lambda}.$$

38. Rewrite the system in the following way

$$a(x+py)+b(x+qy)=ap^2+bq^2$$
$$ap(x+py)+bq(x+qy)=ap^3+bq^3$$
$$\dots\dots\dots\dots\dots\dots$$
$$ap^{k-1}(x+py)+bq^{k-1}(x+qy)=ap^{k+1}+bq^{k+1}.$$

Now it is obvious that the system is equivalent to the following two equations

$$x+py=p^2, \quad x+qy=q^2,$$

and, hence, the system is compatible.

39. We have

$$x_2=a_1-x_1,$$
$$x_3=a_2-x_2=a_2-a_1+x_1,$$
$$x_4=a_3-x_3=a_3-a_2+a_1-x_1,$$
$$\dots\dots\dots\dots\dots\dots$$
$$x_n=a_{n-1}-a_{n-2}+\dots\pm a_2\mp a_1\pm x_1.$$

It should be noted that in the last equality the upper signs will occur when n is odd, and the lower signs when n is even.

Consider the two cases separately.

1° Let n be odd. Then

$$x_n=a_{n-1}-a_{n-2}+\dots+a_2-a_1+x_1.$$

On the other hand,

$$x_n+x_1=a_n.$$

From these two equalities we get

$$x_1=\frac{a_n-a_{n-1}+a_{n-2}-\dots-a_2+a_1}{2},$$

and, hence,

$$x_2=\frac{a_1-a_n+a_{n-1}-\dots-a_3+a_2}{2},$$

$$x_3=\frac{a_2-a_1+a_n-\dots-a_4+a_3}{2},$$

$$\dots\dots\dots\dots\dots\dots$$

2° Let now n be even. Then

$$x_n=a_{n-1}-a_{n-2}+\dots-a_2+a_1-x_1.$$

On the other hand,

$$x_n = a_n - x_1.$$

Consequently, for the given system of equations to be compatible the following equality must be satisfied

$$a_{n-1} - a_{n-2} + \dots - a_2 + a_1 = a_n,$$

i.e.

$$a_n + a_{n-2} + \dots + a_2 = a_{n-1} + a_{n-3} + \dots + a_1$$

(the sum of coefficients with even subscripts must equal the sum of coefficients with odd subscripts). It is apparent that in this case the system will be indeterminate, i.e. will allow an infinite number of solutions, namely:

$$x_1 = \lambda,$$
$$x_2 = a_1 - \lambda,$$
$$x_3 = a_2 - a_1 + \lambda,$$
$$x_4 = a_3 - a_2 + a_1 - \lambda,$$
$$x_n = a_{n-1} - a_{n-2} + \dots + a_3 - a_2 + a_1 - \lambda,$$

where λ is an arbitrary quantity.

40. From the first two equations we find

$$\frac{x}{\frac{b^2}{b-d} - \frac{c^2}{c-d}} = \frac{y}{\frac{c^2}{c-d} - \frac{a^2}{a-d}} = \frac{z}{\frac{a^2}{a-d} - \frac{b^2}{b-d}} = \lambda.$$

Substituting this into the third equation, we have

$$\lambda \left\{ \frac{a}{a-d}\left(\frac{b^2}{b-d} - \frac{c^2}{c-d}\right) + \frac{b}{b-d}\left(\frac{c^2}{c-d} - \frac{a^2}{a-d}\right) + \right.$$
$$\left. + \frac{c}{c-d}\left(\frac{a^2}{a-d} - \frac{b^2}{b-d}\right)\right\} = d(a-b)(b-c)(c-a).$$

After simplification we get

$$\frac{a}{a-d}\left(\frac{b^2}{b-d} - \frac{c^2}{c-d}\right) + \frac{b}{b-d}\left(\frac{c^2}{c-d} - \frac{a^2}{a-d}\right) +$$
$$+ \frac{c}{c-d}\left(\frac{a^2}{a-d} - \frac{b^2}{b-d}\right) = \frac{d(a-b)(b-c)(a-c)}{(a-d)(b-d)(c-d)}.$$

Therefore

$$\lambda = -(a-d)(b-d)(c-d),$$

and, consequently,

$$x = (a - d)(b - c)(db + dc - bc),$$
$$y = (b - d)(c - a)(dc + da - ac),$$
$$z = (c - d)(a - b)(ad + db - ab).$$

41. Solving the last two equations with respect to x and y, we find

$$x + n = \frac{(c-m)(n-a)}{z+c},$$
$$y + b = \frac{(b-l)(m-c)}{z+m}.$$

Hence

$$x + a = \frac{(c-m)(n-a)}{z+c} - (n-a) = (a-n)\frac{z+m}{z+c}.$$

Analogously

$$y + l = (l-b)\frac{z+c}{z+m}.$$

Substituting the found values of $x + a$ and $y + l$ into the first equation, we see that it is a consequence of the two last equtions. Thus, the system is indeterminate, and all its solutions are given by the formulas

$$x = \frac{(c-m)(n-a)}{z+c} - n, \quad y = \frac{(b-l)(m-c)}{z+m} - b,$$

for an arbitrary z.

42. From the second and the third equations we have

$$(1 - k)x + ky = -[(1 + k)x + (12 - k)y],$$

hence, taking into account the first equation, $(5 - k)y = 0$ wherefrom either $k = 5$ or $y = 0$ (hence $x = 0$), which yields (substituting into the second equation) $k = -1$.

43. We have

$$\sin 2a = 2 \sin a \cos a,$$
$$\sin 3a = \sin a\,(4 \cos^2 a - 1),$$
$$\sin 4a = 4 \sin a\,(2 \cos^3 a - \cos a).$$

Therefore the first of the equations of our system is rewritten in the following way

$$x + 2y \cos a + z (4 \cos^2 a - 1) = 4 (2 \cos^3 a - \cos a).$$

The remaining two are similar. Expand this equation in powers of cos a. We have

$$8 \cos^3 a - 4z \cos^2 a - (2y + 4) \cos a + z - x = 0.$$

Putting $\cos a = t$ and dividing both members by 8, we get

$$t^3 - \frac{z}{2} t^2 - \frac{y+2}{4} t + \frac{z-x}{8} = 0. \qquad (*)$$

Our system of equations is equivalent to the statement that the equation (*) has three roots: $t = \cos a$, $t = \cos b$ and $t = \cos c$, wherefrom follows (see Problem 23)

$$\frac{z}{2} = \cos a + \cos b + \cos c,$$
$$\frac{y+2}{4} = -(\cos a \cos b + \cos a \cos c + \cos b \cos c),$$
$$\frac{x-z}{8} = \cos a \cos b \cos c.$$

Therefore the solution of our system will be

$$x = 2 (\cos a + \cos b + \cos c) + 8 \cos a \cos b \cos c,$$
$$y = -2 - 4 (\cos a \cos b + \cos a \cos c + \cos b \cos c),$$
$$z = 2 (\cos a + \cos b + \cos c).$$

44. Put

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k.$$

Since $A + B + C = \pi$, we have

$$\sin A = \sin (B + C) = \sin B \cos C + \cos B \sin C.$$

But from the given proportion we have

$$\sin A = \frac{a}{k}, \quad \sin B = \frac{b}{k}, \quad \sin C = \frac{c}{k}.$$

Substituting this into the last equality, we find

$$a = b \cos C + c \cos B.$$

The rest of the equalities are obtained similarly.

45. Expressing a and b in terms of c and trigonometric functions (from the first two of the given equalities), we get

$$b=\frac{c(\cos A-\cos B\cos C)}{\sin^2 C}, \qquad (1)$$

$$a=\frac{c(\cos B+\cos A\cos C)}{\sin^2 C}. \qquad (2)$$

Substituting (1) and (2) into the third equality and accomplishing all necessary transformations, we find

$$1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=0.$$

Let us now prove that

$$A+B+C=\pi.$$

Transform the obtained equality in the following way

$$\cos^2 A+2\cos A\cos B\cos C=$$
$$=1-\cos^2 B-\cos^2 C-\cos^2 B\cos^2 C+\cos^2 B\cos^2 C,$$
$$\cos^2 A+2\cos A\cos B\cos C+\cos^2 B\cos^2 C=$$
$$=1-\cos^2 B-\cos^2 C(1-\cos^2 B),$$
$$(\cos A+\cos B\cos C)^2=\sin^2 B\sin^2 C.$$

But since we have obtained [see (1)] that

$$\cos A+\cos B\cos C=\frac{b\sin^2 C}{c}>0,$$

we have

$$\cos A+\cos B\cos C=\sin B\sin C,$$
$$\cos A=\sin B\sin C-\cos B\cos C=-\cos(B+C),$$
$$\cos A+\cos(B+C)=2\cos\frac{A+B+C}{2}\cos\frac{A-B-C}{2}=0,$$

wherefrom follows that either

$$\frac{A+B+C}{2}=(2l+1)\frac{\pi}{2}$$

or

$$\frac{A-B-C}{2}=(2l'+1)\frac{\pi}{2},$$

where l and l' are integers. Let us first show that the second case is impossible. In this case we would have

$$A - B - C = (2l' + 1)\pi, \quad B = A - C - (2l' + 1)\pi,$$

$$\cos B = \cos(A - C - \pi) = -\cos(A - C) = $$
$$= -\cos A \cos C - \sin A \sin C.$$

Consequently,

$$\cos B + \cos A \cos C = -\sin A \sin C < 0$$

which is impossible, since we have obtained (2)

$$\cos B + \cos A \cos C = \frac{a \sin^2 C}{c} > 0.$$

Thus, there remains only the case

$$A + B + C = (2l + 1)\pi.$$

However, by virtue of the inequalities, existing for A, B and C, we have

$$0 < 2l + 1 < 3,$$

i.e.

$$2l + 1 = 1$$

and

$$A + B + C = \pi.$$

It only remains to show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

We have shown that

$$\cos A + \cos B \cos C = \sin B \sin C.$$

On the other hand,

$$\cos B + \cos A \cos C = \cos(\pi - A - C) + \cos A \cos C =$$
$$= -\cos(A + C) + \cos A \cos C =$$
$$= \sin A \sin C.$$

Using this equality and also equalities (1) and (2), we easily obtain the required proportion.

46. Let us first show that equation (2) follows from equations (1). Multiplying the first of equations (1) by a, the second by b and the third by $-c$ and adding them termwise we get

$$a^2 + b^2 - c^2 = 2ab \cos C,$$

i.e. the third of equations (2). Likewise we obtain the remaining two of equations (2).

To obtain equations (1) from equations (2) add the first two of (2). Collecting like terms, we find

$$2c^2 - 2\, bc \cos A - 2\, ac \cos B = 0.$$

Hence

$$c = b \cos A + a \cos B,$$

i.e. we get the third of equations (1). The rest of them are obtained similarly.

47. From the first equality we get

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence

$$\sin^2 A = 1 - \cos^2 A =$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} =$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} =$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Consequently

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

Since the given formulas turn one into another by means of a circular permutation of the letters a, b, c, A, B, C, and as a result of this transformation the right member of the last equality remains unchanged, we actually have

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}.$$

But the quantities a, b, c and A, B, C are contained between 0 and π, therefore

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

48. 1° Let us take the last two of the equalities (*) from the preceding problem. We have

$$\cos b - \cos c \cos a = \sin a \sin c \cos B,$$

$$-\cos a \cos b + \cos c = \sin a \sin b \cos C.$$

Multiplying the first of them by $\cos a$ and the second by 1 and then adding, we find

$$-\cos c \cos^2 a + \cos c = \sin a \sin c \cos B \cos a + \sin a \sin b \cos C.$$

Hence

$$\cos c \sin a = \sin c \cos a \cos B + \sin b \cos C.$$

But since it was shown in the preceding problem that from the equalities (*) follows the proportion

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C},$$

in the last equality we can replace the quantities $\sin a$, $\sin b$ and $\sin c$ by ones proportional to them. We get

$$\cos c \sin A = \sin C \cos a \cos B + \sin B \cos C.$$

It is apparent, that there exist six similar equalities. Let us take one more of them, namely, the one which also contains $\cos c$ and $\cos a$. It will have the form

$$\cos a \sin C = \sin A \cos c \cos B + \sin B \cos A.$$

(This equality can be obtained in the following way: multiply the second of the equalities (*) by $\cos c$ and the first one by unity, add them, and in the obtained equality replace $\sin c$ by $\sin C$ and so on.) Thus, we have

$$\cos c \sin A = \sin C \cos a \cos B + \sin B \cos C,$$

$$\cos a \sin C = \sin A \cos c \cos B + \sin B \cos A.$$

Eliminating $\cos c$, we find

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

The rest of the equalities are obtained from this one using a circular permutation.

2° The formulas (*) of Problem 47 make it possible to express $\cos A$, $\cos B$ and $\cos C$ in terms of $\sin a$, $\sin b$,

$\sin c$ and $\cos a$, $\cos b$, $\cos c$. Let us find the expressions for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$. We have

$$2\sin^2\frac{A}{2}=1-\cos A=1-\frac{\cos a-\cos b\cos c}{\sin b\sin c}=$$

$$=\frac{\cos(b-c)-\cos a}{\sin b\sin c},$$

$$2\cos^2\frac{A}{2}=1+\cos A=1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}=$$

$$=\frac{\cos a-\cos(b+c)}{\sin b\sin c}.$$

Hence

$$\sin\frac{A}{2}=\sqrt{\frac{\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}}{\sin b\sin c}},$$

$$\cos\frac{A}{2}=\sqrt{\frac{\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}}{\sin b\sin c}}.$$

Similar expressions are obtained for $\sin\frac{B}{2}$, $\cos\frac{B}{2}$ and $\sin\frac{C}{2}$, $\cos\frac{C}{2}$. Now compute $\sin\frac{A+B}{2}$. We have

$$\sin\frac{A+B}{2}=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}=$$

$$=\sqrt{\frac{\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}}{\sin a\sin b}}\times$$

$$\times\left(\frac{\sin\frac{a+c-b}{2}}{\sin c}+\frac{\sin\frac{b+c-a}{2}}{\sin c}\right)=\cos\frac{C}{2}\cdot\frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}.$$

Thus, we have obtained the following formula

$$\sin\frac{A+B}{2}=\frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}\cos\frac{C}{2}.$$

Likewise we find

$$\cos\frac{A+B}{2}=\frac{\cos\frac{a+b}{2}}{\cos\frac{c}{2}}\sin\frac{C}{2}.$$

Since $\varepsilon=A+B+C-\pi$, we have

$$\frac{A+B}{2}=\frac{\pi}{2}-\frac{C-\varepsilon}{2}.$$

Therefore

$$\sin\frac{A+B}{2}=\cos\frac{C-\varepsilon}{2}$$

and, consequently,

$$\frac{\cos\frac{C-\varepsilon}{2}}{\cos\frac{C}{2}}=\frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}.$$

Hence

$$\frac{\cos\frac{C-\varepsilon}{2}-\cos\frac{C}{2}}{\cos\frac{C-\varepsilon}{2}+\cos\frac{C}{2}}=\frac{\cos\frac{a-b}{2}-\cos\frac{c}{2}}{\cos\frac{a-b}{2}+\cos\frac{c}{2}}$$

and, consequently,

$$\tan\frac{\varepsilon}{4}\tan\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)=\tan\frac{p-b}{2}\tan\frac{p-a}{2}. \qquad (1)$$

Using the formula

$$\cos\frac{A+B}{2}=\frac{\cos\frac{a+b}{2}}{\cos\frac{c}{2}}\sin\frac{C}{2},$$

we find analogously

$$\tan\frac{\varepsilon}{4}\cot\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)=\tan\frac{p}{2}\tan\frac{p-c}{2}. \qquad (2)$$

Multiplying the equalities (1) and (2) termwise and extracting the square root, we get

$$\tan\frac{1}{4}\varepsilon=\sqrt{\tan\frac{p}{2}\tan\frac{p-a}{2}\tan\frac{p-b}{2}\tan\frac{p-c}{2}}.$$

49. We have

$$a[\tan(x+\gamma)-\tan(x+\beta)]+b[\tan(x+\alpha)-\tan(x+\gamma)]+$$
$$+c[\tan(x+\beta)-\tan(x+\alpha)]=0.$$

Hence

$$\frac{a\sin(\gamma-\beta)}{\cos(x+\beta)\cos(x+\gamma)}+\frac{b\sin(\alpha-\gamma)}{\cos(x+\alpha)\cos(x+\gamma)}+$$
$$+\frac{c\sin(\beta-\alpha)}{\cos(x+\beta)\cos(x+\alpha)}=0.$$

$$a\sin(\gamma-\beta)\cos(x+\alpha)+b\sin(\alpha-\gamma)\cos(x+\beta)+$$
$$+c\sin(\beta-\alpha)\cos(x+\gamma)=0.$$

Finally

$$\tan x=\frac{a\sin(\gamma-\beta)\cos\alpha+b\sin(\alpha-\gamma)\cos\beta+c\sin(\beta-\alpha)\cos\gamma}{a\sin(\gamma-\beta)\sin\alpha+b\sin(\alpha-\gamma)\sin\beta+c\sin(\beta-\alpha)\sin\gamma}.$$

50. We have

$$\cos^2\frac{x}{2}=\frac{1}{1+\tan^2\frac{x}{2}}.$$

Therefore

$$\cos x=2\cos^2\frac{x}{2}-1=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

$$\sin x=\tan x\cos x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\cdot\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

It is obvious that if $\tan\frac{x}{2}$ is rational, then $\sin x$ and $\cos x$ are also rational. Show that if $\sin x$ and $\cos x$ are rational, then $\tan\frac{x}{2}$ is rational too.

From the first relationship we have

$$\left(1+\tan^2\frac{x}{2}\right)\cos x=1-\tan^2\frac{x}{2}.$$

Hence

$$\tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}.$$

Consequently, if $\cos x$ is rational, then $\tan^2 \frac{x}{2}$ is rational as well. But from the second equality it follows that

$$2 \tan \frac{x}{2} = \sin x \left(1 + \tan^2 \frac{x}{2}\right).$$

Hence, it is clear that if $\sin x$ and $\cos x$ are rational, then $\tan \frac{x}{2}$ is also rational.

51. Since $\sin^2 x + \cos^2 x = 1$, we have

$$\sin^4 x + \cos^4 x + 2 \sin^2 x \cos^2 x = 1,$$

i.e.

$$\sin^4 x + \cos^4 x = 1 - 2 \sin^2 x \cos^2 x.$$

Therefore the equation is rewritten as

$$1 - 2 \sin^2 x \cos^2 x = a,$$

$$2 \sin^2 x \cos^2 x = 1 - a,$$

$$\sin^2 2x = 2(1 - a), \quad \sin 2x = \pm \sqrt{2(1 - a)}.$$

For the solutions to be real it is necessary and sufficient that

$$\frac{1}{2} \leqslant a \leqslant 1.$$

52. 1° Transforming the left member of the equation, we get

$$\sin x + \sin 3x + \sin 2x = 2 \sin 2x \cos x + \sin 2x =$$
$$= \sin 2x (1 + 2 \cos x) = 0.$$

Hence

$$(1)\ \sin 2x = 0, \quad (2)\ \cos x = -\frac{1}{2}.$$

2° In this case the transformation of the left member yields

$$\cos nx + \cos (n - 2) x - \cos x = 2 \cos (n - 1) x \cos x -$$
$$- \cos x = \cos x [2 \cos (n - 1) x - 1] = 0,$$

i.e. either $\cos x = 0$ or $\cos (n - 1) x = \frac{1}{2}$.

53. 1° We have

$$m(\sin a \cos x - \cos a \sin x) -$$
$$- n(\sin b \cos x - \cos b \sin x) = 0,$$
$$(n \cos b - m \cos a) \sin x - (n \sin b - m \sin a) \cos x = 0,$$
$$(n \cos b - m \cos a) \cos x \left[\tan x - \frac{n \sin b - m \sin a}{n \cos b - m \cos a}\right] = 0.$$

Hence

$$\tan x = \frac{n \sin b - m \sin a}{n \cos b - m \cos a}.$$

2° We have

$$\sin x \cos 3\alpha + \cos x \sin 3\alpha = 3(\sin \alpha \cos x - \cos \alpha \sin x).$$

Hence

$$\sin x(\cos 3\alpha + 3 \cos \alpha) - \cos x(3 \sin \alpha - \sin 3\alpha) = 0.$$

But

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha, \quad \sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha.$$

Therefore the equation takes the form

$$\sin x \cos^3 \alpha - \cos x \sin^3 \alpha = 0.$$

And so

$$\tan x = \tan^3 \alpha.$$

54. It is easy to find that

$$\sin 5x = 16 \sin^5 x - 20 \sin^3 x + 5 \sin x.$$

Therefore our equation takes the form

$$-20 \sin^3 x + 5 \sin x = 0$$

or

$$\sin x(1 - 4 \sin^2 x) = 0.$$

Thus, we have the following solutions

$$\sin x = 0, \quad \sin x = \pm \frac{1}{2}.$$

55. We have

$$2 \sin x \cos(a - x) = \sin a + \sin(2x - a).$$

The equation takes the form

$$\sin x + \sin (2x - a) = 0$$

or

$$2 \sin \frac{3x-a}{2} \cos \frac{x-a}{2} = 0.$$

Thus, the following is possible

$$\sin \frac{3x-a}{2} = 0 \quad \text{and} \quad \frac{3x-a}{2} = k\pi,$$

i.e.

$$3x = a + 2k\pi, \qquad x = \frac{a+2k\pi}{3},$$

where k is any integer.

Similarly, we have

$$\cos \frac{x-a}{2} = 0, \quad \frac{x-a}{2} = (2l+1)\frac{\pi}{2}, \quad x = a + (2l+1)\pi,$$

where l is any integer.

56. We have

$$\sin x \sin (\gamma - x) = \frac{1}{2} [\cos (2x - \gamma) - \cos \gamma].$$

Therefore the equation is rewritten in the following way

$$\cos (2x - \gamma) - \cos \gamma = 2a,$$

$$\cos (2x - \gamma) = 2a + \cos \gamma.$$

57. We have

$$\sin (\alpha + x) + \sin \alpha \sin x \frac{\sin (\alpha + x)}{\cos (\alpha + x)} - m \cos \alpha \cos x = 0.$$

Further

$$\frac{\sin (\alpha + x)}{\cos (\alpha + x)} \{\cos (\alpha + x) + \sin \alpha \sin x\} - m \cos \alpha \cos x = 0.$$

Hence

$$\frac{\sin (\alpha + x)}{\cos (\alpha + x)} \cos \alpha \cos x - m \cos \alpha \cos x =$$

$$= \cos \alpha \cos x \{\tan (\alpha + x) - m\} = 0.$$

Assuming $\cos \alpha \neq 0$, we obtain the following equalities for determining x

$$\cos x = 0, \quad \tan (\alpha + x) = m.$$

58. Rewrite the equation in the following way

$$\cos^2\alpha + \cos^2(\alpha + x) - 2\cos\alpha\cos(\alpha + x) = 1 - \cos^2 x.$$

Hence

$$[\cos\alpha - \cos(\alpha + x)]^2 - \sin^2 x = 0,$$

i.e.

$$[\cos\alpha - \cos(\alpha + x) - \sin x][\cos\alpha - \cos(\alpha + x) + \sin x] = 0.$$

Further

$$[\cos\alpha(1 - \cos x) + \sin x(\sin\alpha - 1)] \times [\cos\alpha(1 - \cos x) + \sin x(\sin\alpha + 1)] = 0,$$

$$\sin^2 x\left[\cos\alpha\tan\frac{x}{2} + \sin\alpha - 1\right] \times \left[\cos\alpha\tan\frac{x}{2} + \sin\alpha + 1\right] = 0$$

(if $\sin x \neq 0$). If $\sin x = 0$, then $\cos^2\alpha(1 - \cos x)^2 = 0$.

Now we easily find the following solutions:

$$\cos x = 1, \quad \tan x = \cot\alpha, \quad \text{i.e. } x = 2k\pi$$

and

$$x = -\alpha + \frac{2k+1}{2}\pi.$$

59. We can readily obtain

$$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}.$$

Therefore

$$(1 - \tan x)\left(1 + \frac{2\tan x}{1 + \tan^2 x}\right) = 1 + \tan x.$$

Hence

$$\frac{(1 - \tan x)(1 + \tan x)^2}{1 + \tan^2 x} - (1 + \tan x) = 0,$$

$$\frac{1 + \tan x}{1 + \tan^2 x}\{1 - \tan^2 x - 1 - \tan^2 x\} = 0,$$

$$\frac{\tan^2 x(1 + \tan x)}{1 + \tan^2 x} = 0.$$

For determining x we have: $\tan x = 0$, $\tan x = -1$.

60. We have

$$\tan A + \tan B = \frac{\sin (A+B)}{\cos A \cos B} .$$

Therefore

$$\tan x + \tan 4x + \tan 2x + \tan 3x = \frac{\sin 5x}{\cos x \cos 4x} +$$
$$+ \frac{\sin 5x}{\cos 2x \cos 3x} = \frac{\sin 5x}{\cos x \cos 2x \cos 3x \cos 4x} \times$$
$$\times \{\cos 2x \cos 3x + \cos x \cos 4x\}.$$

But

$$\cos 3x = 4 \cos^3 x - 3 \cos x.$$

Thus, our equation takes the form

$$\frac{\sin 5x}{\cos 2x \cos 3x \cos 4x} [\cos 2x (4 \cos^2 x - 3) + \cos 4x] = 0.$$

Hence

$$\frac{\sin 5x [4 \cos^2 2x - \cos 2x - 1]}{\cos 2x \cos 3x \cos 4x} = 0.$$

Consequently, either $\sin 5x = 0$, i.e. $5x = k\pi$, or

$$4 \cos^2 2x - \cos 2x - 1 = 0,$$

that is

$$8 \cos 2x = 1 \pm \sqrt{17}.$$

61. Substituting the expressions containing X and Y for x and y into the trinomial

$$ax^2 + 2bxy + cy^2,$$

we get

$$ax^2 + 2bxy + cy^2 = a (X \cos \theta - Y \sin \theta)^2 +$$
$$+ 2b (X \cos \theta - Y \sin \theta) (X \sin \theta + Y \cos \theta) +$$
$$+ c (X \sin \theta + Y \cos \theta)^2 =$$
$$= (a \cos^2 \theta + 2b \cos \theta \sin \theta + c \sin^2 \theta) X^2 +$$
$$+ (a \sin^2 \theta - 2b \sin \theta \cos \theta + c \cos^2 \theta) Y^2 +$$
$$+ (-2a \cos \theta \sin \theta + 2c \cos \theta \sin \theta + 2b \cos^2 \theta -$$
$$-2b \sin^2 \theta) XY.$$

Since, by hypothesis, the coefficient of XY must be equal to zero, we have the following equation for determining θ:

$$2b(\cos^2\theta - \sin^2\theta) - 2(a-c)\sin\theta\cos\theta = 0$$

or

$$2b\cos 2\theta - (a-c)\sin 2\theta = 0.$$

Thus,

$$\tan 2\theta = \frac{2b}{a-c}.$$

62. It is obvious that

$$\frac{x+y}{x-y} = \frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}.$$

Therefore

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta) + \frac{y+z}{y-z}\sin^2(\beta-\gamma) + \frac{z+x}{z-x}\sin^2(\gamma-\alpha) =$$
$$= \sin(2\theta+\alpha+\beta)\sin(\alpha-\beta) + \sin(2\theta+\beta+\gamma)\sin(\beta-\gamma) +$$
$$+ \sin(2\theta+\gamma+\alpha)\sin(\gamma-\alpha).$$

But

$$\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta) = \frac{1}{2}\{\cos(2\theta+2\beta) - \cos(2\theta+2\alpha)\}.$$

Using a circular permutation, we easily check the validity of our identity.

63. 1° Put

$$\frac{\sin x}{a} = \frac{\sin y}{b} = \frac{\sin z}{c} = k.$$

We then have

$$\sin x = ak, \quad \sin y = bk, \quad \sin z = ck.$$

On the other hand,

$$\sin z = \sin(\pi - x - y) = \sin(x+y) =$$
$$= \sin x\cos y + \cos x\sin y.$$

Hence

$$a\cos y + b\cos x = c, \quad b\cos z + c\cos y = a,$$
$$c\cos x + a\cos z = b.$$

Solving this system, we find

$$\cos x = \frac{b^2+c^2-a^2}{2bc}, \quad \cos y = \frac{c^2+a^2-b^2}{2ca},$$

$$\cos z = \frac{a^2+b^2-c^2}{2ab}.$$

At $k = 0$ we get also the following solution $\sin x = \sin y = \sin z = 0$.

2° Put

$$\frac{\tan x}{a} = \frac{\tan y}{b} = \frac{\tan z}{c} = k.$$

Hence

$$\tan x = ak, \quad \tan y = bk, \quad \tan z = ck.$$

Adding these equalities term by term, we get (see Problem 40, Sec. 2)

$$(a+b+c)\,k = \tan x + \tan y + \tan z = \tan x \tan y \tan z.$$

Consequently,

$$(a+b+c)\,k - k^3\,abc = 0.$$

Thus,

$$k = 0, \quad k = \pm\sqrt{\frac{a+b+c}{abc}}.$$

Hence either $\tan x = \tan y = \tan z = 0$ or

$$\tan x = \pm\sqrt{\frac{(a+b+c)\,a}{bc}}, \quad \tan y = \pm\sqrt{\frac{(a+b+c)\,b}{ac}},$$

$$\tan z = \pm\sqrt{\frac{(a+b+c)\,c}{ab}}.$$

64. We have

$$\tan 2b = \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

But, by hypothesis,

$$\tan x \tan y = a,$$

therefore

$$\tan x + \tan y = (1-a)\tan 2b.$$

Knowing the product and sum of the tangents it is easy to find the tangents themselves (see Sec. 5).

65. Transform the equation in the following way

$$4^x + 2^{2x-1} = 3^{x-\frac{1}{2}} + 3^{x+\frac{1}{2}}, \quad 4^x + \frac{1}{2} \cdot 4^x = 3^{x-\frac{1}{2}}(1+3),$$

$$4^x \cdot \frac{3}{2} = 3^{x-\frac{1}{2}} \cdot 4.$$

Hence

$$\frac{4^{x-1}}{2} = 3^{x-\frac{3}{2}}, \quad 2^{2x-3} = (\sqrt{3})^{2x-3}.$$

And so

$$\left(\frac{2}{\sqrt{3}}\right)^{2x-3} = 1.$$

Consequently,

$$2x - 3 = 0 \text{ and } x = \frac{3}{2}.$$

66. Taking logarithms of both members of our equation, we find

$$(x+1)\log_{10} x = 0.$$

Hence

$$x = 1.$$

67. Taking logarithms of the first equation, we find

$$x \log_{10} a + y \log_{10} b = \log_{10} m.$$

Finally, we have to solve the system

$$x \log_{10} a + y \log_{10} b = \log_{10} m,$$
$$x + y = n.$$

68. Put

$$x = b^{\xi}, \quad y = a^{\eta}$$

(from this problem on we assume that $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$ and find positive solutions).

Then (by virtue of the first equation):

$$b^{\xi y} = a^{\eta x}.$$

But

$$b^y = a^x.$$

Consequently,

$$b^{\xi y} = (b^y)^{\xi} = a^{x\xi}.$$

Hence

$$a^{x\xi} = a^{\eta x}, \quad x(\xi - \eta) = 0.$$

Thus, either $x = 0$ or $\eta = \xi$. But at $x = 0$ we get $y = 0$. Rejecting this solution, consider the case $\eta = \xi$.

Consequently,

$$x = b^{\xi} \text{ and } y = a^{\xi}.$$

But

$$x \log a = y \log b,$$

$$b^{\xi} \log a = a^{\xi} \log b, \quad \left(\frac{b}{a}\right)^{\xi} = \frac{\log b}{\log a}.$$

Hence

$$\xi(\log b - \log a) = \log \frac{\log b}{\log a}, \quad \xi = \frac{\log \frac{\log b}{\log a}}{\log b - \log a}.$$

Therefore

$$x = b^{\xi} = \left(b^{\frac{\log \frac{\log b}{\log a}}{\log b}}\right)^{\frac{\log b}{\log b - \log a}}.$$

Since the ratio of logarithms of two numbers is independent of the base chosen, in the expression

$$\frac{\log \frac{\log b}{\log a}}{\log b}$$

we may consider the first logarithms as taken to the base b. Then

$$b^{\frac{\log \frac{\log b}{\log a}}{\log b}} = \frac{\log b}{\log a}$$

and

$$x = \left(\frac{\log b}{\log a}\right)^{\frac{\log b}{\log b - \log a}}$$

Analogously, we find

$$y = \left(\frac{\log b}{\log a}\right)^{\frac{\log a}{\log b - \log a}}$$

69. Taking logarithms of the second equation, we find

$$\frac{\log x}{\log a} = \frac{\log y}{\log b}.$$

Putting this ratio to be equal to ξ, we get

$$x = a^{\xi}, \quad y = b^{\xi}.$$

Substituting these values into the first equation and assuming $a \neq b^{\pm 1}$, we find $\xi = -1$. Thus

$$x=\frac{1}{a}, \quad y=\frac{1}{b}.$$

70. We have

$$x=y^{\frac{x}{y}}.$$

Consequently,

$$x^m=y^{\frac{mx}{y}}.$$

Making use of the second equation, we find

$$y^{\frac{mx}{y}}=y^n.$$

Hence, either $y=1$, and then $x=1$ or $\frac{mx}{y}=n$, i.e.

$$x=\frac{ny}{m}.$$

Substituting into the second equation, we have:

$$\left(\frac{ny}{m}\right)^m=y^n, \quad y^{m-n}=\left(\frac{m}{n}\right)^m.$$

And so

$$y=\left(\frac{m}{n}\right)^{\frac{n}{m-n}}, \quad x=y^{\frac{x}{y}}=\left(\frac{m}{n}\right)^{\frac{m}{m-n}}.$$

SOLUTIONS TO SECTION 5

1. We have

$$x^2\frac{(b+x)(x+c)}{(x-b)(x-c)}=\frac{x^3(b+c+x)+xbcx}{(x-b)(x-c)}.$$

Therefore the left member of our equation is equal to

$$(b+c+x)\left[\frac{x^3}{(x-b)(x-c)}+\frac{b^3}{(b-x)(b-c)}+\frac{c^3}{(c-x)(c-b)}\right]+$$
$$+bcx\left[\frac{x}{(x-b)(x-c)}+\frac{b}{(b-x)(b-c)}+\frac{c}{(c-x)(c-b)}\right].$$

But (see Problem 8, Sec. 2)

$$\frac{x^3}{(x-b)(x-c)}+\frac{b^3}{(b-x)(b-c)}+\frac{c^3}{(c-x)(c-b)}=b+c+x,$$

$$\frac{x}{(x-b)(x-c)}+\frac{b}{(b-x)(b-c)}+\frac{c}{(c-x)(c-b)}=0.$$

Therefore the equation takes the form

$$(b+c+x)^2=(b+c)^2.$$

Hence

$$(b+c+x)^2-(b+c)^2=0,$$

$$(b+c+x-b-c)(b+c+x+b+c)=0,$$

and consequently

$$x_1=0, \quad x_2=-2(b+c).$$

2. Rewrite the equation in the following way

$$(x-a)(x-b)(x-c)(b-c)(c-a)(a-b)\left\{\frac{a^3}{(x-a)(c-a)(a-b)}+\right.$$

$$\left.+\frac{b^3}{(x-b)(b-c)(a-b)}+\frac{c^3}{(x-c)(c-a)(b-c)}\right\}=0.$$

As is known (see Problem 9, Sec. 2)

$$\frac{a^3}{(a-x)(a-b)(a-c)}+\frac{b^3}{(b-x)(b-a)(b-c)}+$$

$$+\frac{c^3}{(c-x)(c-a)(c-b)}+\frac{x^3}{(x-a)(x-b)(x-c)}=1.$$

Therefore, the equation is rewritten as follows

$$(x-a)(x-b)(x-c)(b-c)(c-a)(a-b)\times$$

$$\times\left\{1-\frac{x^3}{(x-a)(x-b)(x-c)}\right\}=0$$

or

$$(b-c)(c-a)(a-b)[(x-a)(x-b)(x-c)-x^3]=0.$$

Assuming that a, b, c are not equal, we get

$$(a+b+c)x^2-(ab+ac+bc)x+abc=0,$$

$$x=\frac{ab+ac+bc\pm\sqrt{(ab+ac+bc)^2-4abc(a+b+c)}}{2(a+b+c)}.$$

For the roots to be equal it is necessary and sufficient that

$$(ab + ac + bc)^2 - 4abc\,(a + b + c) = 0.$$

Hence

$$a^2b^2 + a^2c^2 + b^2c^2 - 2a^2bc - 2b^2ac - 2c^2ab = 0,$$

$$(ab + ac - bc)^2 - 4a^2bc = 0,$$

$$\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{a}\right)^2-\frac{4}{bc}=0.$$

Consequently,

$$\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{a}+\frac{2}{\sqrt{bc}}\right)\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{a}-\frac{2}{\sqrt{bc}}\right)=0$$

or

$$\left[\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{b}}\right)^2-\frac{1}{a}\right]\left[\left(\frac{1}{\sqrt{c}}-\frac{1}{\sqrt{b}}\right)^2-\frac{1}{a}\right]=0.$$

Finally

$$\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}\right)\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)\times$$

$$\times\left(\frac{1}{\sqrt{c}}-\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}\right)\left(\frac{1}{\sqrt{c}}-\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)=0.$$

3. Rewrite the equation in the form

$$\frac{(a-x)^{\frac{3}{2}}+(x-b)^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}+(x-b)^{\frac{1}{2}}}=a-b,$$

wherefrom we have

$$a-x-(a-x)^{\frac{1}{2}}(x-b)^{\frac{1}{2}}+x-b=a-b$$

or

$$\sqrt{(a-x)(x-b)}=0.$$

Thus, the required solutions will be

$$x_1 = a, \quad x_2 = b.$$

4. We have

$$\sqrt{4a+b-5x}+\sqrt{4b+a-5x}=3\sqrt{a+b-2x}.$$

Squaring both members of the equality and performing all the necessary transformations, we get

$$\sqrt{4a+b-5x}\cdot\sqrt{4b+a-5x}=2(a+b-2x).$$

Squaring them once again, we find

$$(4a+b)(4b+a)-5x(4a+b+4b+a)+25x^2 = 4(a^2+b^2+4x^2+2ab-4ax-4bx).$$

Hence

$$x^2-ax-bx+ab=0,$$

and, consequently,

$$x_1=a, \quad x_2=b.$$

Substituting the found values into the original equation, we get

$$\sqrt{b-a}+2\sqrt{b-a}-3\sqrt{b-a}=0.$$

$$2\sqrt{a-b}+\sqrt{a-b}-3\sqrt{a-b}=0.$$

Hence, if $a \neq b$, then the equation has two roots: a and b (strictly speaking, if the operations with complex numbers are regarded as unknown, then there will be only one root).

5. Rewrite the given equation as

$$(1+\lambda)x^2-(a+c+\lambda b+\lambda d)x+ac+\lambda bd=0.$$

Set up the discriminant of this equation $D(\lambda)$. We have

$$D(\lambda)=(a+c+\lambda b+\lambda d)^2-4(1+\lambda)(ac+\lambda bd).$$

On transformation we obtain

$$D(\lambda)=\lambda^2(b-d)^2+2\lambda(ab+ad+bc+dc-2bd-2ac)+(a-c)^2.$$

We have to prove that $D(\lambda) \geqslant 0$ for any λ. Since $D(\lambda)$ is a second-degree trinomial in λ and $D(0)=(a-c)^2>0$, it is sufficient to prove that the roots of this trinomial are imaginary. And for the roots of our trinomial to be imaginary, it is necessary and sufficient that the expression

$$4(ab+ad+bc+dc-2bd-2ac)^2-4(a-c)^2(b-d)^2$$

be less than zero. We have

$$4(ab + ad + bc + dc - 2bd - 2ac)^2 - \\ - 4(a - c)^2(b - d)^2 = \\ = 4(ab + ad + bc + dc - 2bd - 2ac - \\ - ab + cb + ad - cd) \times \\ \times (ab + ad + bc + dc - 2bd - 2ac + ab - \\ - cb - ad + cd) = \\ = -16(b - a)(d - c)(c - b)(d - a).$$

The last expression is really less than zero by virtue of the given conditions

$$a < b < c < d.$$

6. The original equation can be rewritten in the following way

$$3x^2 - 2(a + b + c)x + ab + ac + bc = 0.$$

Let us prove that

$$4(a + b + c)^2 - 12(ab + ac + bc) \geqslant 0.$$

We have

$$4(a + b + c)^2 - 12(ab + ac + bc) = \\ = 4(a^2 + b^2 + c^2 - ab - ac - bc) = \\ = 2(2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc) = \\ = 2\{(a^2 - 2ab + b^2) + (a^2 - 2ac + c^2) + \\ + (b^2 - 2bc + c^2)\} = \\ = 2\{(a - b)^2 + (a - c)^2 + (b - c)^2\} \geqslant 0.$$

7. Suppose the roots of both equations are imaginary. Then

$$p^2 - 4q < 0, \quad p_1^2 - 4q_1 < 0.$$

Consequently

$$p^2 + p_1^2 - 4q - 4q_1 < 0, \quad p^2 + p_1^2 - 2pp_1 < 0, \\ (p - p_1)^2 < 0,$$

which is impossible.

8. Let us rewrite the given equation as

$$(a+b+c)\,x^2 - 2\,(ab+ac+bc)\,x + 3abc = 0.$$

Prove that its discriminant is greater than or equal to zero.
We have

$$4\,(ab+ac+bc)^2 - 12abc\,(a+b+c) =$$
$$= 2\,\{(ab-ac)^2 + (ab-bc)^2 + (ac-bc)^2\} \geqslant 0.$$

9. By properties of the quadratic equation we have the following system

$$p+q=-p, \quad pq=q.$$

From the second equation we get

$$q\,(p-1) = 0.$$

Hence, either $q=0$ or $p=1$. From the first one we find

$$\text{if } q=0, \text{ then } p=0; \text{ if } p=1, \text{ then } q=-2.$$

Thus, we have *two* quadratic equations satisfying the set requirements

$$x^2=0 \text{ and } x^2+x-2=0.$$

10. We have

$$x^2+y^2+z^2-xy-xz-yz =$$
$$= \frac{1}{2}\,(2x^2+2y^2+2z^2-2xy-2xz-2yz) =$$
$$= \frac{1}{2}\,\{(x-y)^2+(x-z)^2+(y-z)^2\} \geqslant 0$$

(see Problems 6 and 8).

But we can reason in a different way. Rearranging our expression in powers of x, we get $x^2-(y+z)\,x+y^2+$ $+z^2-yz$. To prove that this expression is greater than, or equal to, zero for all values of x, it is sufficient to prove that: firstly

$$y^2+z^2-yz \geqslant 0$$

and, secondly,

$$(y+z)^2 - 4\,(y^2+z^2-yz) \leqslant 0.$$

It is evident that there exist the following identities

$$y^2+z^2-yz=\left(y-\frac{1}{2}z\right)^2+\frac{3}{4}z^2,$$

$$(y+z)^2-4(y^2+z^2-yz)=-3(y-z)^2$$

and, consequently, our assertion is proved.

11. We have

$$x^2+y^2+z^2-\frac{a^2}{3}=x^2+y^2+(a-x-y)^2-\frac{a^2}{3}.$$

It is necessary to show that the last expression is greater than, or equal to, zero for all values of x and y. Rearranging this polynomial in powers of y, we get

$$y^2+(x-a)y+x^2-ax+\frac{a^2}{3}.$$

It remains only to prove that for all values of x

$$x^2-ax+\frac{a^2}{3}\geqslant 0, \quad (x-a)^2-4\left(x^2-ax+\frac{a^2}{3}\right)\leqslant 0.$$

We have

$$x^2-ax+\frac{a^2}{3}=\left(x-\frac{a}{2}\right)^2+\frac{1}{12}a^2\geqslant 0,$$

$$(x-a)^2-4\left(x^2-ax+\frac{a^2}{3}\right)=-3\left(x-\frac{1}{3}a\right)^2\leqslant 0,$$

which is the desired result. However, the proof can be carried out in a somewhat different way. Indeed, it is required to prove that

$$3x^2+3y^2+3z^2\geqslant a^2$$

if

$$x^2+y^2+z^2+2xy+2xz+2yz=a^2.$$

Consequently, it suffices to prove that

$$3x^3+3y^2+3z^2\geqslant x^2+y^2+z^2+2xy+2xz+2yz$$

or

$$2x^2+2y^2+2z^2-2xy-2xz-2yz\geqslant 0.$$

And this last inequality is already known to us (see, for instance, Problem 6).

12. See the preceding problem.

13. By the properties of quadratic equation we may write

$$\alpha+\beta=-p, \quad \alpha\beta=q.$$

Therefore

$$s_1 = -p.$$

Since α and β are roots of the equation

$$x^2 + px + q^2 = 0,$$

we have

$$\alpha^2 + p\alpha + q = 0, \quad \beta^2 + p\beta + q = 0.$$

Adding these equalities term by term, we find

$$s_2 + ps_1 + 2q = 0.$$

Hence

$$s_2 = -ps_1 - 2q = p^2 - 2q.$$

Multiplying both members of our equation by x^k, we get

$$x^{k+2} + px^{k+1} + qx^k = 0.$$

Substituting α and β and adding, we find

$$s_{k+2} + ps_{k+1} + qs_k = 0.$$

Putting here $k = 1$, we have

$$s_3 = -ps_2 - qs_1.$$

Further

$$s_3 = -p(p^2 - 2q) + qp = 3pq - p^3$$

Likewise we find

$$s_4 = p^4 - 4p^2q + 2q^2, \quad s_5 = -p^5 + 5p^3q - 5pq^2.$$

To obtain s_{-1}, let us put in our formula $k = -1$. We have

$$s_1 + ps_0 + qs_{-1} = 0.$$

But

$$s_0 = 2, \quad s_1 = -p.$$

Therefore

$$qs_{-1} = +p - 2p = -p, \quad s_{-1} = -\frac{p}{q}.$$

Likewise we get s_{-2}, s_{-3}, s_{-4} and s_{-5}. However, we may proceed as follows

$$s_{-k} = \frac{1}{\alpha^k} + \frac{1}{\beta^k} = \frac{\alpha^k + \beta^k}{(\alpha\beta)^k} = \frac{s_k}{q^k},$$

wherefrom all the desired values of s_{-k} are readily found.

14. Let

$$\sqrt[4]{\alpha}+\sqrt[4]{\beta}=\omega.$$

Then

$$\omega^4=\alpha+4\sqrt[4]{\alpha^3\beta}+6\sqrt[4]{\alpha^2\beta^2}+4\sqrt[4]{\alpha\beta^3}+\beta.$$

But

$$\alpha+\beta=-p, \quad \alpha\beta=q.$$

Consequently

$$\omega^4=-p+6\sqrt{q}+4\sqrt[4]{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta}).$$

But

$$(\sqrt{\alpha}+\sqrt{\beta})^2=\alpha+\beta+2\sqrt{\alpha\beta}=-p+2\sqrt{q},$$

therefore

$$\omega=\sqrt[4]{-p+6\sqrt{q}+4\sqrt[4]{q}\cdot\sqrt{-p+2\sqrt{q}}}.$$

15. Let x be the common root of the given equations. Multiplying the first equation by A', and the second by A and subtracting them termwise, we get

$$(AB'-A'B)x+AC'-A'C=0.$$

Likewise, multiplying the first one by B' and the second by B and subtracting, we find

$$(AB'-A'B)x^2+BC'-B'C=0.$$

Take the value of x from the first obtained equality and substitute it into the second one. Thus, we obtain the required result.

16. Adding all the three equations termwise, we find

$$(x+y+z)^2=a^2+b^2+c^2.$$

Hence

$$x+y+z=\pm\sqrt{a^2+b^2+c^2}.$$

Consequently

$$x=\frac{a^2}{\pm\sqrt{a^2+b^2+c^2}}, \quad y=\frac{b^2}{\pm\sqrt{a^2+b^2+c^2}},$$

$$z=\frac{c^2}{\pm\sqrt{a^2+b^2+c^2}}.$$

17. It is obvious that the system can be rewritten in the following way

$$(x+z)(x+y)=a,$$
$$(y+z)(y+x)=b,$$
$$(z+x)(z+y)=c.$$

Multiplying these equations and extracting a square root from both members of the obtained equality, we have

$$(x+z)(x+y)(y+z)=\pm\sqrt{abc}.$$

Hence

$$y+z=\pm\frac{\sqrt{abc}}{a},\quad x+z=\frac{\pm\sqrt{abc}}{b},\quad x+y=\pm\frac{\sqrt{abc}}{c}.$$

Adding these equalities termwise, we find

$$x+y+z=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

But since

$$y+z=\pm\frac{\sqrt{abc}}{a},$$

we have

$$x=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right).$$

Analogously

$$y=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{a}+\frac{1}{c}-\frac{1}{b}\right),\quad z=\pm\frac{\sqrt{abc}}{2}\left(\frac{1}{b}+\frac{1}{a}-\frac{1}{c}\right),$$

simultaneously taking either pluses or minuses everywhere

18. Put

$$y+x=\gamma,\quad x+z=\beta,\quad y+z=\alpha.$$

Then our equations take the form

$$\gamma+\beta=a\gamma\beta$$
$$\alpha+\gamma=b\alpha\gamma$$
$$\beta+\alpha=c\alpha\beta.$$

Solving this system (see Sec. 4, Problem 17), we find the solutions of the original system

$$x = y = z = 0$$

$$x=\frac{1}{2}\left(\frac{1}{p-b}+\frac{1}{p-c}-\frac{1}{p-a}\right),$$

$$y=\frac{1}{2}\left(\frac{1}{p-c}+\frac{1}{p-a}-\frac{1}{p-b}\right),$$

$$z=\frac{1}{2}\left(\frac{1}{p-a}+\frac{1}{p-b}-\frac{1}{p-c}\right),$$

where

$$2p = a + b + c.$$

19. Adding unity to both members of the equations, we get

$$1 + y + z + yz = a + 1,$$
$$1 + x + z + xz = b + 1,$$
$$1 + x + y + xy = c + 1$$

or

$$(1 + y)(1 + z) = a + 1,$$
$$(1 + x)(1 + z) = b + 1,$$
$$(1 + y)(1 + x) = c + 1.$$

Multiplying these equations, we get

$$(1 + x)^2 (1 + y)^2 (1 + z)^2 = (1 + a)(1 + b)(1 + c)$$

or

$$(1+x)(1+y)(1+z) = \pm\sqrt{(1+a)(1+b)(1+c)}.$$

Consequently,

$$1+x=\pm\sqrt{\frac{(1+b)(1+c)}{1+a}}, \quad 1+y=\pm\sqrt{\frac{(1+a)(1+c)}{1+b}},$$

$$1+z=\pm\sqrt{\frac{(1+a)(1+b)}{1+c}}.$$

20. Multiplying the given equations, we obtain

$$(xyz)^2 = ab\,cx\,yz.$$

First of all we have an obvious solution $x = y = z = 0$. Then

$$xyz = abc.$$

From the original equations we find

$$xyz = ax^2, \quad xyz = by^2, \quad xyz = cz^2.$$

Hence

$$ax^2 = abc, \quad by^2 = abc, \quad cz^2 = abc,$$

$$x^2 = bc, \quad y^2 = ac, \quad z^2 = ab.$$

Thus, we have the following solution set

$$x = \sqrt{bc}, \quad y = \sqrt{ac}, \quad z = \sqrt{ab};$$

$$x = -\sqrt{bc}, \quad y = -\sqrt{ac}, \quad z = \sqrt{ab};$$

$$x = \sqrt{bc}, \quad y = -\sqrt{ac}, \quad z = -\sqrt{ab};$$

$$x = -\sqrt{bc}, \quad y = \sqrt{ac}, \quad z = -\sqrt{ab}.$$

21. Adding the first two equations and subtracting the third one, we get

$$2x^2 = (c + b - a)\, xyz.$$

Likewise we find

$$2y^2 = (c + a - b)\, xyz, \quad 2z^2 = (a + b - c)\, xyz.$$

Singling out the solution

$$x = y = z = 0,$$

we have

$$2x = (c + b - a)\, yz, \quad 2y = (c + a - b)\, xz,$$

$$2z = (a + b - c)\, xy.$$

Then proceed as in the preceding problem.

22. The system is reduced to the form

$$xy + xz = a^2,$$

$$yz + yx = b^2,$$

$$zx + zy = c^2.$$

Adding these equations term by term, we find

$$xy + xz + yz = \frac{1}{2}(a^2 + b^2 + c^2).$$

Taking into consideration the first three equations, we get

$$yz = \frac{b^2 + c^2 - a^2}{2}, \quad zx = \frac{a^2 + c^2 - b^2}{2}, \quad xy = \frac{a^2 + b^2 - c^2}{2}.$$

Multiplying them, we have

$$(xyz)^2 = \frac{(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)}{8},$$

i.e.

$$xyz = \pm\sqrt{\frac{(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)}{8}}.$$

Now we easily find

$$x = \pm\sqrt{\frac{(a^2+c^2-b^2)(a^2+b^2-c^2)}{8(b^2+c^2-a^2)}},$$

$$y = \pm\sqrt{\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)}{8(a^2+c^2-b^2)}},$$

$$z = \pm\sqrt{\frac{(a^2+c^2-b^2)(b^2+c^2-a^2)}{8(a^2+b^2-c^2)}}.$$

23. Adding and subtracting the given equations termwise, we find

$$x^3 + y^3 = a(x+y) + b(x+y) = (a+b)(x+y),$$
$$x^3 - y^3 = a(x-y) - b(x-y) = (a-b)(x-y).$$

Hence

$$(x+y)(x^2 - xy + y^2 - a - b) = 0,$$
$$(x-y)(x^2 + xy + y^2 - a + b) = 0.$$

Thus, we have to consider the following systems

1° $x + y = 0, \quad x - y = 0;$

2° $x + y = 0, \quad x^2 + xy + y^2 - a + b = 0;$

3° $x - y = 0, \quad x^2 - xy + y^2 - a - b = 0;$

4° $x^2 - xy + y^2 - a - b = 0, \ x^2 + xy + y^2 - a + b = 0.$

The first three systems yield the following solutions

1° $x = y = 0;$

2° $x = \pm\sqrt{a-b}, \quad y = \mp\sqrt{a-b};$

3° $x = y = \pm\sqrt{a+b}.$

The last system is reduced to the following one

$$x^2 + y^2 = a, \quad xy = -b.$$

Solving it, we get

$$x = \frac{1}{2}(\varepsilon\sqrt{a-2b} + \eta\sqrt{a+2b}),$$

$$y = \frac{1}{2}(\varepsilon\sqrt{a-2b} - \eta\sqrt{a+2b}),$$

where ε and η take on the values ± 1 independently of each other. Thus, we get four more solutions.

24. Reduce the system to the following form

$$(x + y - z)(x + z - y) = a,$$

$$(y + z - x)(y + x - z) = b,$$

$$(x + z - y)(z + y - x) = c.$$

Multiplying and taking a square root, we get

$$(x + y - z)(x + z - y)(y + z - x) = \pm\sqrt{abc}.$$

Further

$$y + z - x = \pm\sqrt{\frac{bc}{a}},$$

$$x + z - y = \pm\sqrt{\frac{ac}{b}},$$

$$x + y - z = \pm\sqrt{\frac{ab}{c}}.$$

Consequently

$$x = \pm\left(\sqrt{\frac{ac}{b}} + \sqrt{\frac{ab}{c}}\right), \quad y = \pm\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ab}{c}}\right),$$

$$z = \pm\left(\sqrt{\frac{bc}{a}} + \sqrt{\frac{ac}{b}}\right).$$

25. Put

$$\frac{x+y}{x+y+cxy} = \gamma, \quad \frac{y+z}{y+z+ayz} = \alpha, \quad \frac{x+z}{x+z+bxz} = \beta.$$

Then the system takes the form

$$b\gamma + c\beta = a, \quad c\alpha + a\gamma = b, \quad a\beta + b\alpha = c$$

or

$$\frac{\gamma}{c}+\frac{\beta}{b}=\frac{a}{bc}, \quad \frac{\alpha}{a}+\frac{\gamma}{c}=\frac{b}{ac}, \quad \frac{\beta}{b}+\frac{\alpha}{a}=\frac{c}{ab}.$$

Therefore

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=\frac{1}{2}\,\frac{a^2+b^2+c^2}{abc}$$

and, consequently,

$$\alpha=\frac{b^2+c^2-a^2}{2bc}, \quad \beta=\frac{a^2+c^2-b^2}{2ac}, \quad \gamma=\frac{a^2+b^2-c^2}{2ab}.$$

Further

$$\frac{x+y+cxy}{x+y}=\frac{1}{\gamma}, \quad \frac{cxy}{x+y}=\frac{1}{\gamma}-1, \quad \frac{x+y}{cxy}=\frac{\gamma}{1-\gamma}.$$

Finally

$$\frac{1}{x}+\frac{1}{y}=\frac{c\gamma}{1-\gamma}.$$

Analogously, we find

$$\frac{1}{x}+\frac{1}{z}=\frac{b\beta}{1-\beta}, \quad \frac{1}{y}+\frac{1}{z}=\frac{a\alpha}{1-\alpha},$$

wherefrom we find x, y and z.

26. Multiplying the first, second and third equations respectively by y, z and x, we get

$$cx+ay+bz=0.$$

Likewise, multiplying these equations by z, x and y, we find

$$bx+cy+az=0.$$

From these two equations (see Problem 35, Sec. 4) we obtain

$$\frac{x}{a^2-bc}=\frac{y}{b^2-ac}=\frac{z}{c^2-ab}=\lambda,$$

i.e.

$$x=(a^2-bc)\,\lambda, \quad y=(b^2-ac)\,\lambda, \quad z=(c^2-ab)\,\lambda.$$

Substituting these expressions into the third equation, we find

$$\lambda^2=\frac{c}{(c^2-ab)^2-(a^2-bc)(b^2-ac)}=\frac{1}{a^3+b^3+c^3-3abc}.$$

Now it is easy to find x, y and z.

27. Rewrite the system as follows

$$(y^2 - xz) + (z^2 - xy) = a$$
$$(x^2 - yz) + (z^2 - xy) = b$$
$$(x^2 - zy) + (y^2 - zx) = c.$$

Hence

$$x^2 - yz = \frac{b+c-a}{2}, \quad y^2 - xz = \frac{a+c-b}{2}, \quad z^2 - xy = \frac{a+b-c}{2},$$

i.e. we have obtained a system as in the preceding problem

28. Subtracting the equations term by term, we have

$$(x - y)(x + y + z) = b^2 - a^2,$$
$$(x - z)(x + y + z) = c^2 - a^2.$$

Put $x + y + z = t$, then

$$(x - y)\,t = b^2 - a^2, \quad (x - z)\,t = c^2 - a^2.$$

Adding these two equations termwise, we have

$$[3x - (x + y + z)]\,t = b^2 + c^2 - 2a^2.$$

Hence

$$x = \frac{t^2 + b^2 + c^2 - 2a^2}{3t}.$$

Analogously

$$y = \frac{t^2 + a^2 + c^2 - 2b^2}{3t}, \quad z = \frac{t^2 + a^2 + b^2 - 2c^2}{3t}.$$

Substituting these values of x, y and z in one of the equations, we find

$$t^4 - (a^2 + b^2 + c^2)\,t^2 + a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 = 0.$$

Hence

$$t^2 = \frac{a^2 + b^2 + c^2 \pm \sqrt{3(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2}.$$

Knowing t, we obtain the values of x, y and z.

29. We have the following identities

$$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2(xy + xz + yz),$$
$$(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y + z)(xy + xz + yz) - 3xyz.$$

Taking into account the second and third equations of our system, we get from the first identity

$$xy + xz + yz = 0.$$

From the second identity we have

$$xyz = 0.$$

Thus, we obtain the following solutions of our system

$$x = 0, \quad y = 0, \quad z = a;$$
$$x = 0, \quad y = a, \quad z = 0;$$
$$x = a, \quad y = 0, \quad z = 0.$$

30. Let x, y, z and u be the roots of the following fourth-degree equation

$$\alpha^4 - p\alpha^3 + q\alpha^2 - r\alpha + t = 0. \qquad (*)$$

Put

$$x^k + y^k + z^k + u^k = s_k.$$

Then

$$s_4 - ps_3 + qs_2 - rs_1 + t = 0.$$

But by hypothesis

$$s_4 = a^4, \quad s_3 = a^3, \quad s_2 = a^2, \quad s_1 = a.$$

Therefore, the following identity must take place

$$a^4 - pa^3 + qa^2 - ra + t = 0,$$

i.e. the equation (*) has the root $\alpha = a$, and therefore one of the unknowns, say x, is equal to a.

Then there must take place the equalities

$$u + y + z = 0, \quad u^2 + y^2 + z^2 = 0, \quad u^3 + y^3 + z^3 = 0,$$

and, consequently, (by virtue of the results of the last problem)

$$u = y = z = 0.$$

Thus, the given system has the following solutions

$$x = a, \quad u = y = z = 0;$$
$$y = a, \quad x = u = z = 0;$$
$$z = a, \quad x = y = u = 0;$$
$$u = a, \quad x = y = z = 0,$$

31. Equivalence of these systems follows from the identity

$$(a^2 + b^2 + c^2 - 1)^2 + (a'^2 + b'^2 + c'^2 - 1)^2 + \\ + (a''^2 + b''^2 + c''^2 - 1)^2 + 2(aa' + bb' + cc')^2 + \\ + 2(aa'' + bb'' + cc'')^2 + 2(a'a'' + b'b'' + c'c'')^2 = \\ = (a^2 + a'^2 + a''^2 - 1)^2 + (b^2 + b'^2 + b''^2 - 1)^2 + \\ + (c^2 + c'^2 + c''^2 - 1)^2 + 2(ab + a'b' + a''b'')^2 + \\ + 2(ac + a'c' + a''c'')^2 + \\ + 2(bc + b'c' + b''c'')^2.$$

It should be noted that nine coefficients: a, a', a'', b, b', b'', c, c' and c'' can be (as it was established by Euler) expressed in terms of three independent quantities p, q and r in the following way

$$a = \frac{1+p^2-q^2-r^2}{N}, \quad b = \frac{2(r+pq)}{N}, \quad c = \frac{2(-q+pr)}{N},$$

$$a' = \frac{2(-r+pq)}{N}, \quad b' = \frac{1-p^2+q^2-r^2}{N}, \quad c' = \frac{2(p+qr)}{N},$$

$$a'' = \frac{2(q+pr)}{N}, \quad b'' = \frac{2(-p+rq)}{N}, \quad c'' = \frac{1-p^2-q^2+r^2}{N}$$

$$(N = 1 + p^2 + q^2 + r^2).$$

32. Multiplying the first three equalities, we get

$$x^2y^2z^2(y + z)(x + z)(x + y) = a^3b^3c^3.$$

Using the fourth equality, we have

$$(y + z)(x + z)(x + y) = abc$$

or

$$x^2(y + z) + y^2(x + z) + z^2(x + y) + 2xyz = abc.$$

But adding the first three equalities, we find

$$x^2(y + z) + y^2(x + z) + z^2(x + y) = a^3 + b^3 + c^3.$$

Thus, finally

$$a^3 + b^3 + c^3 + abc = 0.$$

33. Adding the three given equalities, we get

$$a + b + c = \frac{(y-z)(z-x)(x-y)}{xyz}.$$

Similarly, we have

$$a-b-c=\frac{(y-z)(z+x)(x+y)}{xyz},$$

$$b-c-a=\frac{(z-x)(x+y)(y+z)}{xyz},$$

$$c-a-b=\frac{(x-y)(y+z)(z+x)}{xyz}.$$

Hence

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)=$$

$$=-\left(\frac{y}{z}-\frac{z}{y}\right)^2\left(\frac{z}{x}-\frac{x}{z}\right)^2\left(\frac{x}{y}-\frac{y}{x}\right)^2=-a^2b^2c^2.$$

Hence, we finally get the result of the elimination

$$2b^2c^2+2b^2a^2+2a^2c^2-a^4-b^4-c^4+a^2b^2c^2=0.$$

34. We have

$$\frac{y}{z}+\frac{z}{y}=2a,\quad \frac{z}{x}+\frac{x}{z}=2b,\quad \frac{x}{y}+\frac{y}{x}=2c.$$

Squaring these equalities and adding them, we get

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+6=4a^2+4b^2+4c^2.$$

On the other hand, multiplying these equalities, we find

$$\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}+2=8abc.$$

Consequently, the result of eliminating x, y and z from the given system is

$$a^2+b^2+c^2-2abc=1.$$

35. We have an identity

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)=$$

$$=4b^2c^2-(b^2+c^2-a^2)^2.$$

Replacing in the right member a^2, b^2 and c^2 by their expressions in terms of x, y and z, and using the relationship

$$xy+xz+yz=0,$$

we get

$$4b^2c^2-(b^2+c^2-a^2)^2=0.$$

Thus, the actual result of eliminating x, y and z from the given system is

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c)=0.$$

36. We have

$$(x+y)^3 = x^3+y^3+3xy(x+y) =$$
$$= x^3+y^3+\frac{3}{2}(x+y)[(x+y)^2-(x^2+y^2)].$$

And so

$$(x+y)^3 = 3(x+y)(x^2+y^2)-2(x^3+y^3).$$

But

$$x+y=a, \quad x^2+y^2=b, \quad x^3+y^3=c.$$

Consequently, the result of the elimination is

$$a^3 = 3ab-2c.$$

37. Put

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{1}{\lambda}.$$

Then

$$a=x\lambda, \quad b=y\lambda, \quad c=z\lambda. \qquad (*)$$

On the other hand, we have

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc.$$

Since $a+b+c=1$, $a^2+b^2+c^2=1$, we obtain from the last equality

$$ab+ac+bc=0.$$

Taking into consideration the equalities (*), we find

$$xy+xz+yz=0.$$

38. We have

$$\left(\alpha-\frac{z}{x}\right)\left(\alpha-\frac{x}{y}\right)\left(\alpha-\frac{y}{z}\right)=\gamma$$

or

$$\alpha^3-\left(\frac{z}{x}+\frac{x}{y}+\frac{y}{z}\right)\alpha^2+\left(\frac{z}{y}+\frac{x}{z}+\frac{y}{x}\right)\alpha-1=\gamma.$$

Hence

$$\alpha\beta-1=\gamma.$$

39. From the first two equalities we find

$$\left.\begin{aligned} z(d-c)+x(d-a)+y(d-b)&=0,\\ w(d-c)+x(a-c)+y(b-c)&=0.\end{aligned}\right\} \qquad (*)$$

Multiplying the first equality by y, and the second by x, and adding them, we get

$$(zy+wx)(d-c)=x^2(c-a)+y^2(b-d)+ \\ +xy(a+c-b-d).$$

We find in the same way that

$$(zx+wy)(d-c)=x^2(a-d)+y^2(c-b)+ \\ +xy(b+c-a-d),$$

$$zw(d-c)^2=x^2(a-d)(c-a)+ \\ +y^2(b-d)(c-b)+ \\ +xy[(a-d)(c-b)+(b-d)(c-a)].$$

Substituting the found expressions for $zy+wx$, $zx+wy$ and zw into the third equality, we get

$$Ax^2+2Bxy+Cy^2=0,$$

where

$$A=(c-a)(a-d)^2(b-c)^2+(c-d)\times \\ \times(b-d)^2(c-a)^2+ \\ +(a-d)(c-a)(d-c)(a-b)^2,$$

$$C=(b-d)(a-d)^2(b-c)^2+ \\ +(c-b)(b-d)^2(c-a)^2+ \\ +(b-d)(c-b)(d-c)(a-b)^2,$$

$$2B=(a+c-b-d)(a-d)^2(b-c)^2+ \\ +(b+c-a-d)(b-d)^2(c-a)^2+ \\ +(d-c)^3(a-b)^2+[(a-d)(c-b)+ \\ +(b-d)(c-a)](d-c)(a-b)^2.$$

Performing all the necessary transformations (the work can be simplified by making use of the result of Problem 8, Sec. 2), we find

$$A = (a-d)^2 (c-a)^2 (c-d),$$
$$B = (d-c)(a-d)(b-c)(a-c)(d-b),$$
$$C = (c-b)^2 (b-d)^2 (c-d).$$

Therefore we have

$$Ax^2 + 2Bxy + Cy^2 = (c-d)\,[(a-d)(a-c)\,x - (b-c)(d-b)\,y]^2 = 0.$$

Hence

$$\frac{x}{(b-c)(d-b)} = \frac{y}{(a-d)(a-c)}.$$

Substituting these values into the equality (*), we get the required proportion.

40. 1° We have

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \left(2\cos^2\frac{\alpha+\beta}{2} - 1\right) = \frac{3}{2}$$

or

$$4\cos^2\frac{\alpha+\beta}{2} - 4\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} + 1 = 0.$$

Hence

$$\cos\frac{\alpha+\beta}{2} = \frac{4\cos\frac{\alpha-\beta}{2} \pm \sqrt{16\cos^2\frac{\alpha-\beta}{2} - 16}}{8}.$$

Since the radicand is equal to $-16\sin^2\frac{\alpha-\beta}{2}$ and $\cos\frac{\alpha+\beta}{2}$ is real, the expression $-16\sin^2\frac{\alpha-\beta}{2}$ must be greater than, or equal to, zero. But this expression cannot exceed zero. Therefore we have

$$\sin\frac{\alpha-\beta}{2} = 0.$$

But since $0 < \alpha < \pi$ and $0 < \beta < \pi$, we have $\alpha = \beta$ and, consequently,

$$\cos\alpha = \frac{1}{2}$$

and

$$\alpha = \beta = \frac{\pi}{3}.$$

2° Analogous to 1°.

41. By hypothesis

$$2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}=a,\quad 2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}=b.$$

Hence

$$\tan\frac{\theta+\varphi}{2}=\frac{b}{a}.$$

But

$$\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},\quad \sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

Therefore

$$\cos(\theta+\varphi)=\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}=\frac{a^2-b^2}{a^2+b^2},\quad \sin(\theta+\varphi)=\frac{2\cdot\frac{b}{a}}{1+\frac{b^2}{a^2}}=\frac{2ab}{a^2+b^2}.$$

42. By hypothesis we have $a\cos\alpha+b\sin\alpha=c$, $a\cos\beta+b\sin\beta=c$. Adding these equalities termwise we find

$$2a\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2b\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2c.$$

Hence

$$\cos\frac{\alpha-\beta}{2}=\frac{c}{a\cos\frac{\alpha+\beta}{2}+b\sin\frac{\alpha+\beta}{2}}=$$

$$=\frac{c}{\cos\frac{\alpha+\beta}{2}\left(a+b\tan\frac{\alpha+\beta}{2}\right)}.$$

Subtracting now the given equalities termwise, we obtain

$$-2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}+2b\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}=0.$$

Since α and β are different solutions of the equation, then $\sin\frac{\alpha-\beta}{2}\neq 0$. Consequently, the last equality yields

$$\tan\frac{\alpha+\beta}{2}=\frac{b}{a}.$$

Let us return to computing $\cos^2\frac{\alpha-\beta}{2}$. We have

$$\cos^2\frac{\alpha-\beta}{2}=\frac{c^2}{\cos^2\frac{\alpha+\beta}{2}\left(a+b\tan\frac{\alpha+\beta}{2}\right)^2}=$$

$$=c^2\left(1+\frac{b^2}{a^2}\right)\frac{1}{\left(a+b\frac{b}{a}\right)^2}=\frac{c^2}{a^2+b^2}.$$

43. Rewrite the given equalities in the following way

$$\sin\theta\,(b\cos\alpha-a\cos\beta)=\cos\theta\,(b\sin\alpha-a\sin\beta),$$
$$\sin\theta\,(d\sin\alpha-c\sin\beta)=\cos\theta\,(c\cos\beta-d\cos\alpha).$$

Eliminating θ, we find

$$(b\cos\alpha-a\cos\beta)(c\cos\beta-d\cos\alpha)=$$
$$=(b\sin\alpha-a\sin\beta)(d\sin\alpha-c\sin\beta).$$

Hence

$$bc\cos\alpha\cos\beta-ac\cos^2\beta-bd\cos^2\alpha+ad\cos\alpha\cos\beta=$$
$$=bd\sin^2\alpha-ad\sin\alpha\sin\beta-bc\sin\alpha\sin\beta+ac\sin^2\beta$$

or

$$(bc+ad)\cos\alpha\cos\beta+(bc+ad)\sin\alpha\sin\beta=bd+ac.$$

Finally

$$\cos(\alpha-\beta)=\frac{bd+ac}{bc+ad}.$$

44. 1° We have

$$\frac{e^2-1}{1+2e\cos\alpha+e^2}=\frac{1+2e\cos\beta+e^2}{e^2-1}=$$
$$=\frac{2e^2+2e\cos\beta}{2e^2+2e\cos\alpha}=\frac{e+\cos\beta}{e+\cos\alpha}$$

(by the property of proportions, from the equality $\frac{a}{b}=\frac{c}{d}$ follows $\frac{a+c}{b+d}=\frac{a}{b}$).

Similarly, we have

$$\frac{e^2-1}{1+2e\cos\alpha+e^2}=\frac{1+2e\cos\beta+e^2}{e^2-1}=$$
$$=\frac{-2-2e\cos\beta}{2+2e\cos\alpha}=-\frac{1+e\cos\beta}{1+e\cos\alpha}.$$

Then

$$\left(\frac{e+\cos\beta}{e+\cos\alpha}\right)^2=\frac{(1+e\cos\beta)^2}{(1+e\cos\alpha)^2}=\frac{e^2+\cos^2\beta-1-e^2\cos^2\beta}{e^2+\cos^2\alpha-1-e^2\cos^2\alpha}=\frac{\sin^2\beta}{\sin^2\alpha}.$$

Consequently,

$$\frac{e^2-1}{1+2e\cos\alpha+e^2}=-\frac{1+e\cos\beta}{1+e\cos\alpha}=\pm\frac{\sin\beta}{\sin\alpha}.$$

2° From the given equality follows (see the result of 1°)

$$\frac{e+\cos\beta}{e+\cos\alpha}=-\frac{1+e\cos\beta}{1+e\cos\alpha}.$$

Consequently,

$$\frac{e+\cos\beta-1-e\cos\beta}{e+\cos\alpha+1+e\cos\alpha}=\frac{e+\cos\beta+1+e\cos\beta}{e+\cos\alpha-1-e\cos\alpha}$$

$\left(\text{from the equality } \frac{a}{b}=\frac{c}{d} \text{ follows } \frac{a+c}{b+d}=\frac{a-c}{b-d}\right).$

Further

$$\frac{(1-e)(1-\cos\beta)}{(1+e)(1+\cos\alpha)}=\frac{(1+e)(1+\cos\beta)}{(1-e)(1-\cos\alpha)}$$

or

$$(1-\cos\beta)(1-\cos\alpha)=\frac{(1+e)^2}{(1-e)^2}(1+\cos\beta)(1+\cos\alpha).$$

Finally

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\pm\frac{1+e}{1-e}.$$

45. Solving the given equation with respect to $\cos x$, we find

$$\cos x(\sin^2\beta\cos\alpha-\sin^2\alpha\cos\beta)=$$
$$=\cos^2\alpha\sin^2\beta-\sin^2\alpha\cos^2\beta=\cos^2\alpha-\cos^2\beta.$$

But

$$\sin^2\beta\cos\alpha-\sin^2\alpha\cos\beta=\cos\alpha(1-\cos^2\beta)-$$
$$-\cos\beta(1-\cos^2\alpha)=\cos\alpha-\cos\beta+$$
$$+\cos\alpha\cos\beta(\cos\alpha-\cos\beta)=$$
$$=(\cos\alpha-\cos\beta)(1+\cos\alpha\cos\beta)$$

therefore

$$\cos x=\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}.$$

Further

$$\tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}=\frac{1+\cos\alpha\cos\beta-\cos\alpha-\cos\beta}{1+\cos\alpha\cos\beta+\cos\alpha+\cos\beta}=$$

$$=\frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$$

and consequently

$$\tan\frac{x}{2}=\pm\tan\frac{\alpha}{2}\tan\frac{\beta}{2}.$$

46. We have

$$\sin^2\alpha=4\sin^2\frac{\varphi}{2}\sin^2\frac{\theta}{2}=(1-\cos\varphi)(1-\cos\theta)=$$

$$=\left(1-\frac{\cos\alpha}{\cos\beta}\right)\left(1-\frac{\cos\alpha}{\cos\gamma}\right).$$

Hence

$$1-\cos^2\alpha=1-\cos\alpha\frac{\cos\beta+\cos\gamma}{\cos\beta\cos\gamma}+\frac{\cos^2\alpha}{\cos\beta\cos\gamma},$$

i.e.

$$\cos^2\alpha\left(1+\frac{1}{\cos\beta\cos\gamma}\right)=\cos\alpha\,\frac{\cos\beta+\cos\gamma}{\cos\beta\cos\gamma}.$$

Assuming that $\cos\alpha$ is nonzero, we find

$$\cos\alpha=\frac{\cos\gamma+\cos\beta}{1+\cos\gamma\cos\beta}.$$

Now it is easy to check that

$$\tan^2\frac{\alpha}{2}=\tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2}.$$

47. Put $\tan\frac{\theta}{2}=\alpha$, $\tan\frac{\theta_1}{2}=\beta$. Then the first two equalities take the form

$$x\alpha^2-2y\alpha+2a-x=0,\quad x\beta^2-2y\beta+2a-x=0.$$

Consequently α and β are the roots of the quadratic equation

$$xz^2-2yz+2a-x=0.$$

Therefore

$$\alpha+\beta=\frac{2y}{x},\quad \alpha\beta=\frac{2a-x}{x}.$$

Furthermore

$$\alpha-\beta=2l.$$

Let us now eliminate α and β from the last three equalities. We have identically

$$(\alpha+\beta)^2=(\alpha-\beta)^2+4\alpha\beta.$$

Consequently,

$$\left(\frac{2y}{x}\right)^2=4l^2+4\,\frac{2a-x}{x}.$$

After simplification we actually get

$$y^2=2ax-(1-l^2)\,x^2.$$

48. From the first two equalities it is obvious that θ and φ are the roots of the equation

$$x\cos\alpha+y\sin\alpha-2a=0\ \text{(unknown } \alpha).$$

It is clear that θ and φ are also the roots of the equation

$$(2a-x\cos\alpha)^2=y^2\sin^2\alpha.$$

Transform the last equation in the following way

$$x^2\cos^2\alpha-4ax\cos\alpha+4a^2=y^2(1-\cos^2\alpha),$$
$$(x^2+y^2)\cos^2\alpha-4ax\cos\alpha+4a^2-y^2=0.$$

Therefore the quantities $\cos\theta$ and $\cos\varphi$ are the roots of the following equation

$$(x^2+y^2)z^2-4axz+4a^2-y^2=0,$$

and therefore

$$\cos\theta\cos\varphi=\frac{4a^2-y^2}{x^2+y^2},\qquad \cos\theta+\cos\varphi=\frac{4ax}{x^2+y^2}.$$

We then have

$$4\sin^2\frac{\theta}{2}\sin^2\frac{\varphi}{2}=4\,\frac{1-\cos\theta}{2}\cdot\frac{1-\cos\varphi}{2}=1$$

or $1-(\cos\theta+\cos\varphi)+\cos\theta\cos\varphi=1$. Hence, $y^2=4a(a-x)$.

49. We have

$$\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}=\frac{\tan^2\frac{\theta}{2}-\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\theta}{2}\tan^2\frac{\alpha}{2}}.$$

But

$$\tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta},$$

$$\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}.$$

Consequently

$$\tan\frac{\theta+\alpha}{2}\cdot\tan\frac{\theta-\alpha}{2}=\frac{\dfrac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}-\dfrac{1-\cos\alpha}{1+\cos\alpha}}{1-\dfrac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}\cdot\dfrac{1-\cos\alpha}{1+\cos\alpha}}=$$

$$=\frac{1-\cos\beta}{1+\cos\beta}=\tan^2\frac{\beta}{2}.$$

50. We have

$$\frac{a+c}{b+d}=\frac{\cos x+\cos(x+2\theta)}{\cos(x+\theta)+\cos(x+3\theta)}=\frac{\cos(x+\theta)\cos\theta}{\cos(x+2\theta)\cos\theta}=\frac{b}{c}.$$

Hence

$$\frac{a+c}{b}=\frac{b+d}{c}.$$

51. We have

$$1+\tan^2\theta=\frac{\cos\beta}{\cos\alpha},\qquad 1+\tan^2\varphi=\frac{\cos\beta}{\cos\gamma}.$$

Hence

$$\frac{\tan^2\theta}{\tan^2\varphi}=\frac{\cos\beta-\cos\alpha}{\cos\alpha}\cdot\frac{\cos\gamma}{\cos\beta-\cos\gamma}.$$

On the other hand, it is given that

$$\frac{\tan^2\theta}{\tan^2\varphi}=\frac{\tan^2\alpha}{\tan^2\gamma}.$$

Therefore we have

$$\frac{\cos\beta-\cos\alpha}{\cos\beta-\cos\gamma}\cdot\frac{\cos\gamma}{\cos\alpha}=\frac{\tan^2\alpha}{\tan^2\gamma}.$$

From this equality we get

$$\cos\beta=\frac{\cos^2\alpha\sin^2\gamma-\cos^2\gamma\sin^2\alpha}{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma}=\frac{\sin^2\gamma-\sin^2\alpha}{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma}.$$

But

$$\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta}=\frac{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma-\sin^2\gamma+\sin^2\alpha}{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma+\sin^2\gamma-\sin^2\alpha}=$$

$$=\frac{\sin^2\alpha(1-\cos\gamma)-\sin^2\gamma(1-\cos\alpha)}{\sin^2\gamma(1+\cos\alpha)-\sin^2\alpha(1+\cos\gamma)}=$$

$$=\frac{8\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2}-8\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\sin^2\frac{\alpha}{2}}{8\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2}-8\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2}}=$$

$$=\frac{\sin^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2}\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{\gamma}{2}\right)}{\cos^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2}\left(\sin^2\frac{\gamma}{2}-\sin^2\frac{\alpha}{2}\right)}=\tan^2\frac{\alpha}{2}\cdot\tan^2\frac{\gamma}{2},$$

since

$$\cos^2\frac{\alpha}{2}-\cos^2\frac{\gamma}{2}=\sin^2\frac{\gamma}{2}-\sin^2\frac{\alpha}{2}.$$

52. Put

$$\tan\frac{\theta}{2}=x,\qquad \tan\frac{\varphi}{2}=y.$$

Then

$$\cos\theta=\frac{1-x^2}{1+x^2}=\cos\alpha\cos\beta,\qquad \cos\varphi=\frac{1-y^2}{1+y^2}=$$

$$=\cos\alpha_1\cdot\cos\beta.$$

Further

$$x^2=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta},\qquad y^2=\frac{1-\cos\alpha_1\cos\beta}{1+\cos\alpha_1\cos\beta},$$

therefore

$$\tan^2\frac{\beta}{2}=x^2y^2=\frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha_1\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha_1\cos\beta)}.$$

Add unity to both members of the equality. We find

$$\frac{2}{1+\cos\beta}=\frac{2(1+\cos\alpha\cos\alpha_1\cos^2\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha_1\cos\beta)}.$$

Assuming $\cos\beta\neq 0$, we obtain

$$\cos\alpha+\cos\alpha_1=1+\cos\alpha\cos\alpha_1\cos^2\beta,$$

i.e.

$$\cos\alpha + \cos\alpha_1 = 1 + \cos\alpha\cos\alpha_1 (1 - \sin^2\beta),$$

$$\cos\alpha\cos\alpha_1\sin^2\beta = 1 + \cos\alpha\cos\alpha_1 - \cos\alpha - \cos\alpha_1 = (1 - \cos\alpha)(1 - \cos\alpha_1),$$

and, consequently, indeed

$$\sin^2\beta = \left(\frac{1}{\cos\alpha} - 1\right)\left(\frac{1}{\cos\alpha_1} - 1\right).$$

53. We have

$$\frac{\cos(\beta-\gamma)-\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\beta+\gamma)} = \frac{\cos(\gamma-\alpha)-\cos(\beta-\gamma)}{\cos(\beta+\gamma)-\cos(\gamma+\alpha)} = \frac{\cos(\alpha-\beta)-\cos(\gamma-\alpha)}{\cos(\gamma+\alpha)-\cos(\alpha+\beta)} = x.$$

Hence

$$\frac{\sin\left(\frac{\alpha+\gamma}{2}-\beta\right)}{\sin\left(\frac{\alpha+\gamma}{2}+\beta\right)} = \frac{\sin\left(\frac{\beta+\alpha}{2}-\gamma\right)}{\sin\left(\frac{\beta+\alpha}{2}+\gamma\right)} = \frac{\sin\left(\frac{\gamma+\beta}{2}-\alpha\right)}{\sin\left(\frac{\gamma+\beta}{2}+\alpha\right)}$$

or

$$\frac{\tan\beta-\tan\frac{\alpha+\gamma}{2}}{\tan\beta+\tan\frac{\alpha+\gamma}{2}} = \frac{\tan\gamma-\tan\frac{\beta+\alpha}{2}}{\tan\gamma+\tan\frac{\beta+\alpha}{2}} = \frac{\tan\alpha-\tan\frac{\beta+\gamma}{2}}{\tan\alpha+\tan\frac{\beta+\gamma}{2}}$$

But from the equalities

$$\frac{a-b}{a+b} = \frac{a'-b'}{a'+b'} = \frac{a''-b''}{a''+b''}$$

follows

$$\frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''}.$$

Therefore we have

$$\frac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)} = \frac{\tan\beta}{\tan\frac{1}{2}(\alpha+\gamma)} = \frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}.$$

54. From the first equality we have

$$\frac{(\tan\theta\cos\beta-\sin\beta)\cos\alpha}{(\tan\varphi\cos\alpha-\sin\alpha)\cos\beta} + \frac{(\cos\alpha-\tan\theta\sin\alpha)\sin\beta}{(\cos\beta+\tan\varphi\sin\beta)\sin\alpha} = 0.$$

Hence

$$\sin\alpha\cos\beta\cos(\alpha-\beta)\tan\theta+\sin\beta\cos\alpha\cos(\alpha+\beta)\tan\varphi=$$
$$=2\sin\beta\cos\beta\sin\alpha\cos\alpha. \qquad (*)$$

From the second equality we get

$$\frac{\tan\theta}{\tan\varphi}=-\frac{\cos(\alpha-\beta)\tan\beta}{\cos(\alpha+\beta)\tan\alpha}.$$

Therefore we may put

$$\tan\theta=\lambda\cos(\alpha-\beta)\tan\beta,$$
$$\tan\varphi=-\lambda\cos(\alpha+\beta)\tan\alpha.$$

Substituting the expressions for $\tan\theta$ and $\tan\varphi$ into the equality $(*)$, we find

$$\lambda=\frac{1}{2\sin\alpha\sin\beta}.$$

Thus

$$\tan\theta=\frac{\cos(\alpha-\beta)}{2\sin\alpha\cos\beta}=\frac{1}{2}(\cot\alpha+\tan\beta),$$

$$\tan\varphi=-\frac{\cos(\alpha+\beta)}{2\cos\alpha\sin\beta}=\frac{1}{2}(\tan\alpha-\cot\beta).$$

55. We have

$$\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos(\alpha-\beta)=$$
$$=\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta-$$
$$-2\sin^2\alpha\sin^2\beta=\sin^2\alpha-\sin^2\alpha\sin^2\beta+$$
$$+\sin^2\beta-\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta=$$
$$=\sin^2\alpha\cos^2\beta+\sin^2\beta\cos^2\alpha-$$
$$-2\sin\alpha\sin\beta\cos\alpha\cos\beta=$$
$$=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2=\sin^2(\alpha-\beta).$$

Therefore

$$\sin(\alpha-\beta)=\pm n\sin(\alpha+\beta),$$
$$\sin\alpha\cos\beta-\cos\alpha\sin\beta=\pm n(\sin\alpha\cos\beta+\cos\alpha\sin\beta),$$
$$\tan\alpha-\tan\beta=\pm n(\tan\alpha+\tan\beta).$$

Finally

$$\tan\alpha=\frac{1\pm n}{1\mp n}\tan\beta.$$

56. Expanding the given equalities, we get

$$\cos\alpha\cos 3\theta + \sin\alpha\sin 3\theta = m\cos^3\theta,$$
$$\sin\alpha\cos 3\theta - \cos\alpha\sin 3\theta = m\sin^3\theta.$$

Multiplying the first equality by $\cos 3\theta$, the second by $-\sin 3\theta$ and adding them term by term, we find

$$\cos\alpha = m\{\cos^3\theta\cos 3\theta - \sin^3\theta\sin 3\theta\}.$$

But it is known that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Consequently

$$\cos^3\theta\cos 3\theta - \sin^3\theta\sin 3\theta = 4(\cos^6\theta + \sin^6\theta) - 3(\sin^4\theta + \cos^4\theta).$$

But squaring the original equality and adding, we get

$$\cos^6\theta + \sin^6\theta = \frac{1}{m^2}.$$

Compute $\cos^4\theta + \sin^4\theta$. We have

$$\cos^6\theta + \sin^6\theta = (\cos^2\theta + \sin^2\theta)(\cos^4\theta + \sin^4\theta - \cos^2\theta\sin^2\theta) = \cos^4\theta + \sin^4\theta - \cos^2\theta\sin^2\theta.$$

Therefore

$$\frac{1}{m^2} = (\cos^2\theta + \sin^2\theta)^2 - 3\sin^2\theta\cos^2\theta,$$
$$3\sin^2\theta\cos^2\theta = 1 - \frac{1}{m^2},$$
$$\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{2}{3}\left(1 - \frac{1}{m^2}\right) = \frac{1}{3}\left(1 + \frac{2}{m^2}\right).$$

Thus

$$\cos\alpha = m\{4(\cos^6\theta + \sin^6\theta) - 3(\sin^4\theta + \cos^4\theta)\} = m\left\{\frac{4}{m^2} - 1 - \frac{2}{m^2}\right\} = \frac{2 - m^2}{m},$$

i.e.

$$m^2 + m\cos\alpha = 2.$$

57. From the first equality we obtain

$$a[\sin(\theta + \varphi) - \sin(\theta - \varphi)] = b[\sin(\theta - \varphi) + \sin(\theta + \varphi)].$$

Hence

$$a \tan \varphi = b \tan \theta.$$

Consequently

$$\frac{a}{b} \tan \varphi = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}.$$

But from the second equality we have

$$\tan \frac{\theta}{2} = \frac{b \tan \frac{\varphi}{2} + c}{a},$$

therefore

$$\frac{a}{b} \frac{2 \tan \frac{\varphi}{2}}{1 - \tan^2 \frac{\varphi}{2}} = \frac{2\left(b \tan \frac{\varphi}{2} + c\right)}{a\left[1 - \frac{\left(b \tan \frac{\varphi}{2} + c\right)^2}{a^2}\right]}.$$

Putting for brevity $\tan \frac{\varphi}{2} = x$ and transforming the last equality, we find

$$bc(1 + x^2) = -(b^2 + c^2 - a^2)\, x.$$

But

$$\frac{2x}{1 + x^2} = \sin \varphi.$$

Finally

$$\sin \varphi = \frac{2bc}{a^2 - b^2 - c^2}.$$

58. From the third equality we obtain

$$\sin^2 \theta \sin^2 \varphi = (\cos \theta \cos \varphi - \sin \beta \sin \gamma)^2.$$

Using the first two equalities, we find

$$\left(1 - \frac{\sin^2 \beta}{\sin^2 \alpha}\right)\left(1 - \frac{\sin^2 \gamma}{\sin^2 \alpha}\right) = \left(\frac{\sin \beta \sin \gamma}{\sin^2 \alpha} - \sin \beta \sin \gamma\right)^2.$$

After some transformations this equality yields

$$\tan^2 \alpha = \tan^2 \gamma + \tan^2 \beta.$$

59. We have

$$a \sin^2 \theta + b \cos^2 \theta = 1, \quad a \cos^2 \varphi + b \sin^2 \varphi = 1.$$

Hence

$$a\tan^2\theta + b = 1 + \tan^2\theta, \quad b\tan^2\varphi + a = 1 + \tan^2\varphi.$$

Consequently

$$(a-1)\tan^2\theta = 1 - b, \quad (b-1)\tan^2\varphi = 1 - a,$$

$$\frac{\tan^2\theta}{\tan^2\varphi} = \left(\frac{1-b}{1-a}\right)^2.$$

On the other hand,

$$\frac{\tan^2\theta}{\tan^2\varphi} = \frac{b^2}{a^2}.$$

From the last two equalities we get (assuming that a is not equal to b)

$$a + b - 2ab = 0.$$

60. Rewrite the first two equalities in the following way

$$\cos\theta\cos\alpha + \sin\theta\sin\alpha = a, \quad \sin\theta\cos\beta - \cos\theta\sin\beta = b.$$

Multiplying first the former by $\sin\beta$ and the latter by $\cos\alpha$, and then the former by $\cos\beta$ and the latter by $-\sin\alpha$ and adding them, we find

$$\sin\theta\cos(\alpha-\beta) = a\sin\beta + b\cos\alpha,$$
$$\cos\theta\cos(\alpha-\beta) = a\cos\beta - b\sin\alpha.$$

Squaring the last two equalities and adding them, we get

$$\cos^2(\alpha-\beta) = a^2 - 2ab\sin(\alpha-\beta) + b^2.$$

61. Since

$$\cos 3x = \cos^3 x - 3\sin^2 x\cos x,$$
$$\sin 3x = -\sin^3 x + 3\sin x\cos^2 x,$$

the equation takes the form

$$(\cos^3 x - 3\sin^2 x\cos x)\cos^3 x + (-\sin^3 x + 3\sin x\cos^2 x)\sin^3 x = 0,$$

$$\cos^6 x - 3\cos^4 x\sin^2 x + 3\sin^4 x\cos^2 x - \sin^6 x = 0$$

or

$$(\cos^2 x - \sin^2 x)^3 = 0, \quad \cos 2x = 0.$$

62. Since

$$\sin 2x + 1 = (\sin x + \cos x)^2,$$

we have

$$(\sin x + \cos x)^2 + (\sin x + \cos x) + \cos^2 x - \sin^2 x = 0.$$

Hence

$$(\sin x + \cos x)(1 + 2\cos x) = 0$$

or

$$\cos x\,(1 + \tan x)\,(1 + 2\cos x) = 0.$$

And so

$$\tan x = -1 \text{ and } \cos x = -\frac{1}{2}$$

are the required solutions of our equation.

63. We have

$$\frac{\sin^2 x}{\cos^2 x} - \frac{1-\cos x}{1-\sin x} = 0.$$

Hence

$$\frac{(\cos^3 x - \sin^3 x) - (\cos^2 x - \sin^2 x)}{\cos^2 x\,(1-\sin x)} = 0$$

or

$$(1 - \tan x)(1 - \cos x) = 0.$$

Hence

$$\tan x = 1 \text{ and } \cos x = 1.$$

64. We have

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

Therefore

$$\cos 6x = 4\cos^3 2x - 3\cos 2x.$$

On the other hand,

$$\cos^6 x = \left(\frac{1+\cos 2x}{2}\right)^3.$$

The equation takes the following form

$$4(1 + \cos 2x)^3 - (4\cos^3 2x - 3\cos 2x) = 1$$

or

$$4\cos^2 2x + 5\cos 2x + 1 = 0.$$

Thus

$$\cos 2x = -1, \quad \cos 2x = -\frac{1}{4}.$$

65. We have

$$\sin 2x\cos x + \cos 2x\sin x + \sin 2x - m\sin x = 0.$$

Hence

$$\sin x\,[2\cos^2 x + \cos 2x + 2\cos x - m] = 0,$$
$$\sin x\,[4\cos^2 x + 2\cos x - (m + 1)] = 0.$$

And so, one solution is

$$\sin x = 0.$$

The other is obtained by the formula

$$\cos x = \frac{-1 \pm \sqrt{4m+5}}{4}.$$

Hence, first of all, it follows that there must be

$$4m + 5 \geqslant 0.$$

Further, for one of the roots to exist it is required that $|-1 + \sqrt{4m+5}| \leqslant 4$, i.e. that $-4 \leqslant -1 + \sqrt{4m+5} \leqslant$ $\leqslant +4$ or $-3 \leqslant \sqrt{4m+5} \leqslant 5$, i.e. $m \leqslant 5$. For the other root to exist it is necessary that

$$|-1 - \sqrt{4m+5}| \leqslant 4, \quad -4 \leqslant -1 - \sqrt{4m+5} \leqslant 4,$$
$$m \leqslant 1$$

Thus if $m < -\frac{5}{4}$, then $\cos x$ has no real values; at $m = -\frac{5}{4}$ it has one real value $\left(\cos x = -\frac{1}{4}\right)$; for $-\frac{5}{4} < m \leqslant 1$ $\cos x$ has two real values $\left(\cos x = \frac{-1 \pm \sqrt{4m+5}}{4}\right)$ and for $1 <$ $< m \leqslant 5$ $\cos x$ again has one real value $\left(\cos x = \right.$ $\left.= \frac{-1+\sqrt{4m+5}}{4}\right)$ and at $m > 5$ it has no real values.

66. Rewrite the equation as

$$\frac{1}{\cos(x-\alpha)} \{(1+k)\cos x \cos(2x-\alpha) -$$
$$-(1+k\cos 2x)\cos(x-\alpha)\} = 0.$$

But

$$\cos x \cos(2x-\alpha) = \frac{1}{2}\cos(3x-\alpha) + \frac{1}{2}\cos(x-\alpha),$$

$$\cos 2x \cos(x-\alpha) = \frac{1}{2}\cos(3x-\alpha) + \frac{1}{2}\cos(x+\alpha).$$

Therefore

$$\frac{1}{\cos(x-\alpha)}\{(1+k)[\cos(3x-\alpha)+\cos(x-\alpha)]-$$
$$-2\cos(x-\alpha)-k[\cos(3x-\alpha)+\cos(x+\alpha)]\}=0$$

or

$$\frac{1}{\cos(x-\alpha)}\{\cos(3x-\alpha)-\cos(x-\alpha)+$$
$$+k[\cos(x-\alpha)-\cos(x+\alpha)]\}=0,$$
$$\frac{\sin x}{\cos(x-\alpha)}\{k\sin\alpha-\sin(2x-\alpha)\}=0.$$

Hence

$$\sin x=0 \text{ and } \sin(2x-\alpha)=k\sin\alpha.$$

67. Since $\sin^2 x+\cos^2 x=1$, we have $\sin^4 x+\cos^4 x+$ $+2\sin^2 x\cos^2 x=1$ and $\sin^4 x+\cos^4 x=1-\frac{1}{2}(\sin 2x)^2$. The equation takes the following form

$$\sin^2 2x-8\sin 2x+4=0.$$

Hence

$$\sin 2x=4\pm\sqrt{16-4},\quad \sin 2x=4\pm 2\sqrt{3}.$$

Rejecting one of the solutions, we get finally

$$\sin 2x=4-2\sqrt{3}.$$

68. We have

$$\log_x a=\frac{1}{\log_a x},\quad \log_{ax} a=\frac{1}{\log_a ax},\quad \log_{a^2x} a=\frac{1}{\log_a a^2x}.$$

The equation takes the form

$$\frac{2}{\log_a x}+\frac{1}{\log_a x+1}+\frac{3}{\log_a x+2}=0.$$

Put

$$\log_a x=z.$$

Finally, we have to solve the following equation

$$\frac{2}{z}+\frac{1}{z+1}+\frac{3}{z+2}=0.$$

Hence

$$\frac{6z^2+11z+4}{z(z+1)(z+2)}=0.$$

The required roots are

$$z_1 = -\frac{4}{3}, \quad z_2 = -\frac{1}{2}.$$

Thus

$$x_1 = a^{-\frac{4}{3}}, \quad x_2 = a^{-\frac{1}{2}}.$$

69. We have

$$x = y^{\frac{a}{x+y}}.$$

Hence

$$y^{x+y} = y^{\frac{4a^2}{x+y}}.$$

Consequently, either $y = 1$ or $x + y = \frac{4a^2}{x+y}$. But at $y = 1$ $x^{4a} = 1$ and, consequently, $x = 1$. Thus, we get one solution

$$x = 1, \quad y = 1.$$

Let us now find a second solution. We have

$$(x + y)^2 = 4a^2,$$

i.e.

$$x + y = 2a.$$

Therefore

$$x^{2a} = y^a, \quad \left(\frac{x^2}{y}\right)^a = 1,$$

and consequently

$$x^2 = y,$$

i.e.

$$x^2 = 2a - x.$$

From this quadratic equation we find

$$x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2a}.$$

The positive solution is

$$x = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2a}.$$

The corresponding value of y is found by the formula

$$y = x^2.$$

70. Raising the first equation to the power q and the second to p, we obtain

$$u^{pq}v^{q^2}=a^{xq}, \quad u^{pq}v^{p^2}=a^{yp}.$$

Dividing one of these equalities by the other termwise, we find

$$v^{q^2-p^2}=a^{xq-yp},$$

and consequently

$$v=a^{\frac{py-qx}{p^2-q^2}}.$$

Analogously, we find

$$u=a^{\frac{xp-yq}{p^2-q^2}}. \qquad (*)$$

Substituting these expressions for u and v into the third and fourth equations, we have

$$a^{p(x^2+y^2)-2xyq}=b^{p^2-q^2}, \quad a^{2xyp-q(x^2+y^2)}=c^{p^2-q^2}.$$

Hence

$$p(x^2+y^2)-2xyq=(p^2-q^2)\log_a b,$$

$$2xyp-q(x^2+y^2)=(p^2-q^2)\log_a c.$$

Consequently

$$x^2+y^2=p\log_a b+q\log_a c, \quad 2xy=q\log_a b+p\log_a c;$$

wherefrom we find x and y, and then u and v using the formulas (*).

SOLUTIONS TO SECTION 6

1. Let $x=\alpha+\beta i$, $y=\gamma+\delta i$. Then

$$x+y=\alpha+\gamma+(\beta+\delta)i, \quad x-y=\alpha-\gamma+(\beta-\delta)i,$$

$$|x+y|^2+|x-y|^2=(\alpha+\gamma)^2+(\beta+\delta)^2+$$

$$+(\alpha-\gamma)^2+(\beta-\delta)^2=$$

$$=2(\alpha^2+\beta^2)+2(\gamma^2+\delta^2)=2\{|x|^2+|y|^2\}.$$

2. Let $x = \alpha + \beta i$, hence $\bar{x} = \alpha - \beta i$.

1° By hypothesis,

$$\alpha - \beta i = \alpha^2 - \beta^2 + 2\alpha\beta i.$$

Hence

$$\alpha = \alpha^2 - \beta^2, \quad -\beta = 2\alpha\beta.$$

Therefore

$$\beta(2\alpha + 1) = 0, \quad \alpha = \alpha^2 - \beta^2.$$

Assume first $\beta = 0$, $\alpha = \alpha^2$ or $\alpha(\alpha - 1) = 0$. And so, first of all we have the following solutions

$$\alpha = 0; \quad \beta = 0, \quad x = 0;$$

$$\alpha = 1, \quad \beta = 0, \quad x = 1.$$

Let us now pass over to the case when $2\alpha + 1 = 0$, i.e.

$$\alpha = -\frac{1}{2}, \quad -\frac{1}{2} = \frac{1}{4} - \beta^2, \quad \beta^2 = \frac{3}{4}, \quad \beta = \pm\frac{\sqrt{3}}{2},$$

i.e.

$$x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad x = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Consequently, there exist four complex values of x satisfying the condition

$$\bar{x} = x^2,$$

namely

$$x = 0, \quad x = 1, \quad x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad x = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

2° Let us solve the following system

$$\alpha(\alpha^2 - 3\beta^2 - 1) = 0, \quad \beta(3\alpha^2 - \beta^2 + 1) = 0.$$

We find the following solutions

$$\alpha = 0, \quad \beta = 0;$$

$$\alpha = 0, \quad \beta = \pm 1;$$

$$\alpha = \pm 1, \quad \beta = 0.$$

And so

$$x = 0, \quad x = \pm 1, \quad x = \pm i.$$

3. Put

$$a_1 + b_1 i = x, \quad a_2 + b_2 i = y, \quad \ldots, \quad a_{n-1} + b_{n-1} i = u,$$
$$a_n + b_n i = w.$$

Then the inequality to be proved may be rewritten as

$$|x+y+\ldots+u+w| \leqslant$$
$$\leqslant |x|+|y|+\ldots+|u|+|w|,$$

i.e. we have to prove that the modulus of a sum of several complex numbers is less than or equal to the sum of moduli of the addends. Let us first prove this for two addends, i.e. let us prove that

$$|x+y| \leqslant |x|+|y|.$$

But

$$|x+y| = \sqrt{(a_1+a_2)^2+(b_1+b_2)^2},$$
$$|x| = \sqrt{a_1^2+b_1^2}, \quad |y| = \sqrt{a_2^2+b_2^2}.$$

Consequently, it is required to prove that

$$\sqrt{(a_1+a_2)^2+(b_1+b_2)^2} \leqslant \sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}.$$

On squaring both members of this inequality and after some simplifications we get an equivalent inequality

$$a_1a_2+b_1b_2 \leqslant \sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}.$$

This inequality is undoubtedly true if

$$(a_1a_2+b_1b_2)^2 \leqslant (a_1^2+b_1^2)(a_2^2+b_2^2),$$

i.e. if

$$(a_1a_2+b_1b_2)^2-(a_1^2+b_1^2)(a_2^2+b_2^2) \leqslant 0,$$
$$-(a_1b_2-a_2b_1)^2 \leqslant 0,$$

which is obvious. Thus, it is proved that

$$|x+y| \leqslant |x|+|y|$$

for *any* complex x and y. To prove our proposition for the general case proceed as follows. We have

$$|x+y+z+\ldots+u+w| =$$
$$= |(x+y+\ldots+u)+w| \leqslant |x+y+\ldots+$$
$$+u|+|w|.$$

Let us now apply an analogous operation to the first term

$$|x+y+\ldots+u|.$$

Continuing this operation, we shall prove our proposition for the case of n terms. The above proof was carried out by the method of mathematical induction. Let us add to it another proof. Suppose the complex numbers are reduced to the trigonometric form, i.e. put

$$x=\rho_1(\cos\varphi_1+i\sin\varphi_1),$$

$$y=\rho_2(\cos\varphi_2+i\sin\varphi_2),\ \ldots,\quad w=\rho_n(\cos\varphi_n+i\sin\varphi_n).$$

We then have

$$x+y+\ldots+w=\sum_{k=1}^{n}\rho_k\cos\varphi_k+i\sum_{k=1}^{n}\rho_k\sin\varphi_k,$$

$$|x|+|y|+\ldots+|w|=\sum_{k=1}^{n}\rho_k,$$

$$|x+y+\ldots+w|^2=\Big(\sum_{k=1}^{n}\rho_k\cos\varphi_k\Big)^2+\Big(\sum_{k=1}^{n}\rho_k\sin\varphi_k\Big)^2.$$

It is required to prove that

$$\Delta=\Big(\sum_{k=1}^{n}\rho_k\Big)^2-\Big(\sum_{k=1}^{n}\rho_k\cos\varphi_k\Big)^2-\Big(\sum_{k=1}^{n}\rho_k\sin\varphi_k\Big)^2\geqslant 0.$$

we have

$$\Big(\sum_{k=1}^{n}\rho_k\Big)^2=\sum_{k=1}^{n}\rho_k^2+2\sum_{s\neq t}\rho_s\rho_t,$$

$$\Big(\sum_{k=1}^{n}\rho_k\cos\varphi_k\Big)^2=\sum_{k=1}^{n}\rho_k^2\cos^2\varphi_k+2\sum_{s\neq t}\rho_s\rho_t\cos\varphi_s\cos\varphi_t,$$

$$\Big(\sum_{k=1}^{n}\rho_k\sin\varphi_k\Big)^2=\sum_{k=1}^{n}\rho_k^2\sin^2\varphi_k+2\sum_{s\neq t}\rho_s\rho_t\sin\varphi_s\sin\varphi_t,$$

consequently

$$\Delta=2\sum_{s\neq t}\rho_s\rho_t-2\sum_{s\neq t}\rho_s\rho_t\cos(\varphi_s-\varphi_t),$$

$$\Delta=2\sum_{s\neq t}\rho_s\rho_t\{1-\cos(\varphi_s-\varphi_t)\}=4\sum_{s\neq t}\rho_s\rho_t\sin^2\frac{\varphi_s-\varphi_t}{2}\geqslant 0.$$

4. Proved by a direct check, taking into consideration that $\varepsilon^2=-\varepsilon-1$, $\varepsilon^3=1$.

5. It is obvious that

$$a^2 + b^2 + c^2 - ab - ac - bc = (a + \varepsilon b + \varepsilon^2 c)(a + \varepsilon^2 b + \varepsilon c),$$

$$x^2 + y^2 + z^2 - xy - xz - yz = (x + \varepsilon y + \varepsilon^2 z)(x + \varepsilon^2 y + \varepsilon z).$$

Therefore

$$(a^2 + b^2 + c^2 - ab - ac - bc)(x^2 + y^2 + z^2 - xy - xz - yz) = [(ax + cy + bz) + (cx + by + az)\varepsilon + (bx + ay + cz)\varepsilon^2] \times [(ax + cy + bz) + (cx + by + az)\varepsilon^2 + (bx + ay + cz)\varepsilon] = X^2 + Y^2 + Z^2 - XY - XZ - YZ,$$

where

$$X = ax + cy + bz, \quad Y = cx + by + az, \quad Z = bx + ay + cz.$$

6. 1° Solving the given system with respect to x, y and z, we get

$$x = \frac{A + B + C}{3}, \quad y = \frac{A + B\varepsilon^2 + C\varepsilon}{3}, \quad z = \frac{A + B\varepsilon + C\varepsilon^2}{3}.$$

2° We have

$$|A|^2 + |B|^2 + |C|^2 = A\bar{A} + B\bar{B} + C\bar{C}.$$

But

$$A\bar{A} = (x + y + z)(\bar{x} + \bar{y} + \bar{z}) = |x|^2 + |y|^2 + |z|^2 + \bar{x}(y + z) + \bar{y}(x + z) + \bar{z}(x + y),$$

$$B\bar{B} = (x + y\varepsilon + z\varepsilon^2)(\bar{x} + \bar{y}\varepsilon^2 + \bar{z}\varepsilon) = |x|^2 + |y|^2 + |z|^2 + \bar{x}(y\varepsilon + z\varepsilon^2) + \bar{y}(x\varepsilon^2 + z\varepsilon) + \bar{z}(x\varepsilon + y\varepsilon^2),$$

$$C\bar{C} = (x + y\varepsilon^2 + z\varepsilon)(\bar{x} + \bar{y}\varepsilon + \bar{z}\varepsilon^2) = |x|^2 + |y|^2 + |z|^2 + \bar{x}(y\varepsilon^2 + z\varepsilon) + \bar{y}(x\varepsilon + z\varepsilon^2) + \bar{z}(x\varepsilon^2 + y\varepsilon).$$

Adding the three equalities term by term, we find

$$|A|^2 + |B|^2 + |C|^2 = A\bar{A} + B\bar{B} + C\bar{C} =$$
$$= 3\,[|x|^2 + |y|^2 + |z|^2] + x\,[\bar{y}\,(1 + \varepsilon + \varepsilon^2) +$$
$$+ z\,(1 + \varepsilon^2 + \varepsilon)] + \bar{y}\,[x\,(1 + \varepsilon^2 + \varepsilon) +$$
$$+ z\,(1 + \varepsilon + \varepsilon^2)] + \bar{z}\,[x\,(1 + \varepsilon + \varepsilon^2) +$$
$$+ y\,(1 + \varepsilon^2 + \varepsilon)].$$

But since $1 + \varepsilon + \varepsilon^2 = 0$, the last three expressions in square brackets are equal to zero and

$$|A|^2 + |B|^2 + |C|^2 = 3\,[|x|^2 + |y|^2 + |z|^2].$$

7. On the basis of the result obtained in 1° of Problem 6, we have

$$x'' = \frac{AA' + BB' + CC'}{3}, \quad y'' = \frac{AA' + BB'\varepsilon^2 + CC'\varepsilon}{3},$$
$$z'' = \frac{AA' + BB'\varepsilon + CC'\varepsilon^2}{3}.$$

Further

$$AA' + BB' + CC' = (x + y + z)(x' + y' + z') +$$
$$+ (x + y\varepsilon + z\varepsilon^2)(x' + y'\varepsilon + z'\varepsilon^2) +$$
$$+ (x + y\varepsilon^2 + z\varepsilon)(x' + y'\varepsilon^2 + z'\varepsilon) =$$
$$= 3\,(xx' + zy' + yz').$$

And so $x'' = xx' + zy' + yz'$. Analogously $y'' = yy' + xz' + zx'$, $z'' = zz' + yx' + xy'$ (the last two expressions emerge from the first one as a result of a circular permutation).

8. Though this formula was already proved (see Problem 2, Sec. 1), we are going to demonstrate here another proof, using this time complex numbers.

We have the identity

$$(\alpha\delta - \beta\gamma)(\alpha'\delta' - \beta'\gamma') = (\alpha\alpha' + \beta\gamma')(\gamma\beta' + \delta\delta') -$$
$$- (\alpha\beta' + \beta\delta')(\gamma\alpha' + \delta\gamma'),$$

let us put here

$$\alpha = x + yi, \quad \beta = z + ti, \quad \gamma = -(z - ti), \quad \delta = x - yi.$$
$$\alpha' = a + bi, \quad \beta' = c + di, \quad \gamma' = -(c - di), \quad \delta' = a - bi.$$

Then

$$\alpha\delta - \beta\gamma = x^2 + y^2 + z^2 + t^2,$$

$$\alpha'\delta' - \beta'\gamma' = a^2 + b^2 + c^2 + d^2,$$

$$\alpha\alpha' + \beta\gamma' = (ax - by - cz - dt) + \\ + i(bx + ay + dz - ct),$$

$$\gamma\beta' + \delta\delta' = \overline{\beta\gamma'} + \overline{\alpha\alpha'} = \overline{(\alpha\alpha' + \beta\gamma')}.$$

Therefore

$$(\alpha\alpha' + \beta\gamma')(\gamma\beta' + \delta\delta') = (ax - by - cz - dt)^2 + \\ + (bx + ay + dz - ct)^2.$$

Further

$$\alpha\beta' + \beta\delta' = (cx - dy + az + bt) + \\ + i(dx + cy - bz + at),$$

$$\gamma\alpha' + \delta\gamma' = -(cx - dy + az + bt) + \\ + i(dx + cy - bz + at),$$

i.e.

$$-(\alpha\beta' + \beta\delta')(\gamma\alpha' + \delta\gamma') = (cx - dy + az + bt)^2 + \\ + (dx + cy - bz + at)^2.$$

Substituting the obtained expressions into the original identity, we find

$$(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + t^2) = \\ = (ax - by - cz - dt)^2 + (bx + ay + dz - ct)^2 + \\ + (cx - dy + az + bt)^2 + (dx + cy - bz + at)^2.$$

Replacing in it d by $-d$ and t by $-t$, we get the required identity.

9. Expand the expression $(\cos\varphi + i\sin\varphi)^n$, by the binomial formula. We have

$$(\cos\varphi + i\sin\varphi)^n = \cos^n\varphi + n\cos^{n-1}\varphi\, i\sin\varphi + \\ + \frac{n(n-1)}{1\cdot 2}\cos^{n-2}\varphi\,(i\sin\varphi)^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos^{n-3}\varphi\times \\ \times(i\sin\varphi)^3 + \ldots + n\cos\varphi\,(i\sin\varphi)^{n-1} + (i\sin\varphi)^n.$$

Separating the real part from the imaginary one in this expansion, and using de Moivre's formula, we find

$$\cos n\varphi + i \sin n\varphi = \left(\cos^n \varphi - \frac{n(n-1)}{1\cdot 2}\cos^{n-2}\varphi \sin^2\varphi + \dots\right) + i\left(n\cos^{n-1}\varphi \sin\varphi - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos^{n-3}\varphi\sin^3\varphi + \dots\right).$$

Hence

$$\cos n\varphi = \cos^n\varphi - \frac{n(n-1)}{1\cdot 2}\cos^{n-2}\varphi\sin^2\varphi + \dots,$$

$$\sin n\varphi = n\cos^{n-1}\varphi\sin\varphi - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos^{n-3}\varphi\sin^3\varphi + \dots .$$

Taking into account the parity of n and dividing both members of these equalities by $\cos^n\varphi$, we get the required formulas.

10. First prove case 1°. We have

$$\cos\varphi = \frac{(\cos\varphi + i\sin\varphi) + (\cos\varphi - i\sin\varphi)}{2}.$$

Put $\cos\varphi + i\sin\varphi = \varepsilon$. Then $\cos\varphi - i\sin\varphi = \varepsilon^{-1}$,

$$\cos^{2m}\varphi = \left(\frac{\varepsilon+\varepsilon^{-1}}{2}\right)^{2m} = \frac{1}{2^{2m}}\sum_{k=0}^{2m} C_{2m}^k \varepsilon^{-k}\cdot\varepsilon^{2m-k}.$$

Further

$$2^{2m}\cos^{2m}\varphi = \sum_{k=0}^{m-1} C_{2m}^k\varepsilon^{2(m-k)} + C_{2m}^m + \sum_{k=m+1}^{2m} C_{2m}^k\varepsilon^{2(m-k)}.$$

In the second sum put $m-k = -(m-k')$. Then this sum is rewritten in the following manner.

$$\sum_{k'=m-1}^{0} C_{2m}^{2m-k'}\varepsilon^{-2(m-k')} = \sum_{k=0}^{m-1} C_{2m}^k\varepsilon^{-2(m-k)}.$$

And so

$$2^{2m}\cos^{2m}\varphi = \sum_{k=0}^{m-1} C_{2m}^k\left(\varepsilon^{2(m-k)} + \varepsilon^{-2(m-k)}\right) + C_{2m}^m.$$

However,

$$\varepsilon^{2(m-k)} + \varepsilon^{-2(m-k)} = 2\cos 2(m-k).$$

Therefore,

$$2^{2m}\cos^{2m}\varphi=\sum_{k=0}^{m-1}2C_{2m}^{k}\cos 2(m-k)\varphi+C_{2m}^{m}.$$

Replacing in this formula φ by $\frac{\pi}{2}-\varphi$, we get formula 2°. Formulas 3° and 4° are deduced as 1° and 2°.

11. Form the expression

$$u_n+iv_n=(\cos\alpha+i\sin\alpha)+ \\ +r[\cos(\alpha+\theta)+i\sin(\alpha+\theta)]+\ldots+ \\ +r^n[\cos(\alpha+n\theta)+i\sin(\alpha+n\theta)]= \\ =(\cos\alpha+i\sin\alpha)\{1+r(\cos\theta+i\sin\theta)+\ldots+ \\ +r^n(\cos n\theta+i\sin n\theta)\}.$$

Put

$$\cos\theta+i\sin\theta=\varepsilon.$$

Then

$$u_n+iv_n=(\cos\alpha+i\sin\alpha)\{1+r\varepsilon+\ldots+(r\varepsilon)^n\}= \\ =(\cos\alpha+i\sin\alpha)\frac{(r\varepsilon)^{n+1}-1}{r\varepsilon-1}.$$

Let us transform the fraction $\frac{(r\varepsilon)^{n+1}-1}{r\varepsilon-1}$, separating the real part from the imaginary one.

We have

$$\frac{(r\varepsilon)^{n+1}-1}{r\varepsilon-1}=\frac{[(r\varepsilon)^{n+1}-1][r\bar{\varepsilon}-1]}{(r\varepsilon-1)(r\bar{\varepsilon}-1)}= \\ =\frac{r^{n+2}[\cos n\theta+i\sin n\theta]-r[\cos\theta-i\sin\theta]}{1-2r\cos\theta+r^2}+ \\ +\frac{-r^{n+1}[\cos(n+1)\theta+i\sin(n+1)\theta]+1}{1-2r\cos\theta+r^2}.$$

Multiplying the last fraction by $\cos\alpha+i\sin\alpha$ and separating the real and imaginary parts, we get the required result

$$u_n+iv_n=\frac{r^{n+2}[\cos(n\theta+\alpha)+i\sin(n\theta+\alpha)]}{1-2r\cos\theta+r^2}+ \\ +\frac{-r[\cos(\alpha-\theta)+i\sin(\alpha-\theta)]}{1-2r\cos\theta+r^2}+ \\ +\frac{-r^{n+1}\{\cos[(n+1)\theta+\alpha]+i\sin[(n+1)\theta+\alpha]\}+\cos\alpha+i\sin\alpha}{1-2r\cos\theta+r^2}.$$

Hence

$$u_n = \frac{\cos\alpha - r\cos(\alpha-\theta) - r^{n+1}\cos[(n+1)\theta+\alpha] + r^{n+2}\cos(n\theta+\alpha)}{1-2r\cos\theta+r^2},$$

$$v_n = \frac{\sin\alpha - r\sin(\alpha-\theta) - r^{n+1}\sin[(n+1)\theta+\alpha] + r^{n+2}\sin(n\theta+\alpha)}{1-2r\cos\theta+r^2}.$$

Putting in these formulas $\alpha=0$, $r=1$, we find

$$1+\cos\theta+\cos 2\theta+\ldots+\cos n\theta = \frac{\sin\frac{n+1}{2}\theta\cos\frac{n}{2}\theta}{\sin\frac{\theta}{2}},$$

$$\sin\theta+\sin 2\theta+\ldots+\sin n\theta = \frac{\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}.$$

12. We have

$$S+S'i = \sum_{k=0}^{n} C_n^k(\cos k\theta + i\sin k\theta) = \sum_{k=0}^{n} C_n^k(\cos\theta+i\sin\theta)^k =$$

$$= (1+\cos\theta+i\sin\theta)^n = \left[2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^n =$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n =$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right).$$

Hence

$$S = 2^n\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}, \quad S' = 2^n\cos^n\frac{\theta}{2}\sin\frac{n\theta}{2}.$$

13. Put

$$S = \sin^{2p}\alpha+\sin^{2p}2\alpha+\ldots+\sin^{2p}n\alpha = \sum_{l=1}^{n}\sin^{2p}l\alpha.$$

But (see Problem 10)

$$\sin^{2p}l\alpha = \frac{1}{2^{2p-1}}(-1)^p\sum_{k=0}^{p-1}(-1)^k C_{2p}^k\cos 2(p-k)l\alpha + \frac{1}{2^{2p}}C_{2p}^p,$$

therefore

$$S = \frac{(-1)^p}{2^{2p-1}}\sum_{k=0}^{p-1}(-1)^k C_{2p}^k\sum_{l=1}^{n}\cos 2(p-k)l\alpha + \frac{n}{2^{2p}}C_{2p}^p.$$

Put $2(p-k)\alpha=\xi$. Then

$$\sum_{l=1}^{n}\cos 2(p-k)l\alpha=\cos\xi+\ldots+\cos n\xi=\frac{\sin\frac{n\xi}{2}\cos\frac{n+1}{2}\xi}{\sin\frac{\xi}{2}}$$

(see the solution of Problem 11).

Let us denote

$$\frac{\sin\frac{n\xi}{2}\cos\frac{n+1}{2}\xi}{\sin\frac{\xi}{2}}=\sigma_k.$$

Then we can prove that $\sigma_k=0$ if k is of the same parity as p $\{k\equiv p \pmod 2\}$ and $\sigma_k=-1$ if k and p are of different parity $\{k\equiv p+1 \pmod 2\}$, and we get

$$S=\frac{(-1)^{p+1}}{2^{2p-1}}\sum_{\substack{k=0\\ k\equiv p+1 \pmod 2}}^{p-1}(-1)^k C_{2p}^k+n\frac{1}{2^{2p}}C_{2p}^p.$$

Hence

$$S=\frac{1}{2^{2p-1}}\sum_{\substack{k=0\\ k\equiv p+1 \pmod 2}}^{p-1}C_{2p}^k+n\frac{1}{2^{2p}}C_{2p}^p.$$

But we can prove that $\sum\limits_{\substack{k=0\\ k\equiv p+1 \pmod 2}}^{p-1}C_{2p}^k=2^{2p-2}$ (see Problem 58 of this section) and our formula is deduced.

14. 1° Rewrite the polynomial as

$$x^n-a^n-nxa^{n-1}+na^n=(x^n-a^n)-na^{n-1}(x-a)=$$
$$=(x-a)(x^{n-1}+ax^{n-2}+\ldots+a^{n-1}-na^{n-1}).$$

At $x=a$ the second factor of the last product vanishes and, consequently, is divisible by $x-a$; therefore the given polynomial is divisible by $(x-a)^2$.

2° Let us denote the polynomial by P_n and set up the difference P_n-P_{n-1}. Transforming this difference, we easily prove that it is divisible by $(1-x)^3$. Since it is true for any positive n, we obtain a number of equalities

$$P_n - P_{n-1} = (1-x)^3 \varphi_n(x),$$
$$P_{n-1} - P_{n-2} = (1-x)^3 \varphi_{n-1}(x),$$
$$\dots\dots\dots\dots\dots$$
$$P_3 - P_2 = (1-x)^3 \varphi_2(x),$$
$$P_2 - P_1 = (1-x)^3 \varphi_1(x),$$

where $\varphi_i(x)$ are polynomials with respect to x.

Hence

$$P_n - P_1 = (1-x)^3 \psi(x).$$

But since

$$P_1 = (1-x)^3,$$

it follows that P_n is divisible by $(1-x)^3$ and our proposition is proved.

15. 1° Considering the given expression as a polynomial in y, let us put $y = 0$. We see that at $y = 0$ the polynomial vanishes (for any x). Therefore our polynomial is divisible by y. Since it is symmetrical both with respect to x and y (remains unchanged on permutation of these letters), it is divisible by x as well. Thus, the polynomial is divisible by xy. To prove that it is divisible by $x + y$, let us put in it $y = -x$. It is evident that for odd n we have

$$(x - x)^n - x^n - (-x)^n = 0.$$

Consequently, our polynomial is divisible by $x + y$. It only remains to prove the divisibility of the polynomial by

$$x^2 + xy + y^2 = (y - x\varepsilon)(y - x\varepsilon^2),$$

where

$$\varepsilon^2 + \varepsilon + 1 = 0.$$

For this purpose it only remains to replace y first by $x\varepsilon$ and then by $x\varepsilon^2$ and to make sure that with these substitutions the polynomial vanishes. Since, by hypothesis, n is not divisible by three, it follows that $n = 3l + 1$ or $3l + 2$. At $y = x\varepsilon$ the polynomial attains the following value

$$(x + x\varepsilon)^n - x^n - (x\varepsilon)^n = x^n\{\varepsilon^{2n} + 1 + \varepsilon^n\} = x^n(1 + \varepsilon + \varepsilon^2) = 0.$$

Likewise we prove that at $y = x\varepsilon^2$ the polynomial vanishes as well, and, consequently, its divisibility by $xy(x + y) \times (x^2 + xy + y^2)$ is proved.

2° To prove this statement let us proceed as follows. Let the quantities $-x$, $-y$ and $x+y$ be the roots of a cubic equation

$$\alpha^3 - r\alpha^2 - p\alpha - q = 0.$$

Then, by virtue of the known relations between the roots of an equation and its coefficients (see the beginning of this section), we have

$$r = -x - y + (x + y) = 0, \quad -p = xy - x(x + y) - \\ - y(x + y),$$

$$q = xy(x + y).$$

Thus, $-x, -y$ and $x+y$ are the roots of the following equation

$$\alpha^3 - p\alpha - q = 0,$$

where

$$p = x^2 + xy + y^2, \quad q = xy(x + y).$$

Put

$$(-x)^n + (-y)^n + (x+y)^n = S_n.$$

Between successive values of S_n there exist the following relationships

$$S_{n+3} = pS_{n+1} + qS_n,$$

S_1 being equal to zero. Let us prove that S_n is divisible by p^2 if $n \equiv 1 \pmod 6$ using the method of mathematical induction. Suppose S_n is divisible by p^2 and prove that then S_{n+6} is also divisible by p^2. We have

$$S_{n+6} = pS_{n+4} + qS_{n+3}, \quad S_{n+4} = pS_{n+2} + qS_{n+1}.$$

Therefore

$$S_{n+6} = p(pS_{n+2} + qS_{n+1}) + q(pS_{n+1} + qS_n) = \\ = p^2S_{n+2} + 2pqS_{n+1} + q^2S_n.$$

Since, by supposition, S_n is divisible by p^2, it suffices to prove that S_{n+1} is divisible by p. Thus, we only have to prove that

$$(x+y)^n + (-x)^n + (-y)^n$$

is divisible by $x^2 + xy + y^3$ if $n \equiv 2 \pmod 6$. Proceeding in the same way as in 1°, we easily prove our assertion. And so, assuming that S_n is divisible by p^2, we have proved that S_{n+6} is also divisible by p^2. But $S_1 = 0$ is divisible by p^2. Consequently,

$$S_n = (x+y)^n - x^n - y^n$$

is divisible by $(x^2 + xy + y^2)$ at any $n \equiv 1 \pmod 6$. It only remains to prove its divisibility by $x + y$ and by xy.

16. Equality 1° is obvious. From Problem 15 it follows that $(x + y)^5 - x^5 - y^5$ is divisible by $xy(x + y)(x^2 + xy + y^2)$. Since both the polynomials $(x + y)^5 - x^5 - y^5$ and $xy(x + y)(x^2 + xy + y^2)$ are homogeneous with respect to x and y of one and the same power, the quotient of division $(x + y)^5 - x^5 - y^5$ by $xy(x + y)(x^2 + xy + y^2)$ will be a certain quantity independent of x and y. Let us denote it by A. We then have

$$(x + y)^5 - x^5 - y^5 = Ay(x + y)(x^2 + xy + y^2).$$

Since this equality represents an identity and, hence, holds for all values of x and y, let us put here, for instance, $x = 1$, $y = 1$. We get

$$2^5 - 1 - 1 = A \cdot 2 \cdot 3.$$

Hence $A = 5$, and we finally get

$$(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2).$$

Using the result of Problem 15 (2°), we can write similarly

$$(x + y)^7 - x^7 - y^7 = Axy(x + y)(x^2 + xy + y^2)^2.$$

Putting here $x = y = 1$, we find $A = 7$.

17. It is known that

$$(x + y + z)^3 - x^3 - y^3 - z^3 = 3(x + y)(x + z)(y + z).$$

Let us prove that $(x + y + z)^m - x^m - y^m - z^m$ is divisible by $x + y$. Considering our polynomial rearranged in powers of x, we put in it $x = -y$. We have

$$(-y + y + z)^m - (-y)^m - y^m - z^m = 0,$$

since m is odd.

Consequently, our polynomial is divisible by $(x + y)$. Likewise we make sure that it is divisible by $(x + z)$ and by $(y + z)$.

18. The condition necessary and sufficient for a polynomial $f(x)$ to be divisible by $x - a$ consists in that $f(a) =$

$= 0$. Put

$$f(x) = x^3 + kyzx = y^3 + z^3.$$

For this polynomial to be divisible by $x + y + z$ it is necessary and sufficient that

$$f(-y - z) = 0.$$

However

$$f(-y - z) = -(y + z)^3 - kyz(y + z) + y^3 + z^3 = \\ = -(k + 3)\, yz\,(y + z),$$

wherefrom follows $k = -3$. Thus, for $x^3 + y^3 + z^3 + kxyz$ to be divisible by $x + y + z$ it is necessary and sufficient that $k = -3$.

19. Divide n by p. We get $n = lp + r$, where l is a positive integer and $0 \leqslant r < p$. Consequently,

$$x^n - a^n = x^{lp}x^r - a^{lp}a^r = x^{lp}x^r - a^{lp}x^r + a^{lp}x^r - a^{lp}a^r = \\ = x^r(x^{lp} - a^{lp}) + a^{lp}(x^r - a^r).$$

But $x^{lp} - a^{lp} = (x^p)^l - (a^p)^l$ is divisible by $x^p - a^p$, therefore for the divisibility $x^n - a^n$ by $x^p - a^p$ it is necessary and sufficient that $x^r - a^r$ is divisible by $x^p - a^p$. But it is possible only when $r = 0$, and, consequently, $n = lp$. Finally, for $x^n - a^n$ to be divisible by $x^p - a^p$ it is necessary and sufficient that n is divisible by p.

20. Put $f(x) = x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$. On the other hand,

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1) = \\ = (x + 1)(x + i)(x - i).$$

It only remains to show that

$$f(-1) = f(i) = f(-i) = 0.$$

21. We have

$$1 + x^2 + x^4 + \ldots + x^{2n-2} = \frac{x^{2n} - 1}{x^2 - 1},$$

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

It is required to find out at what n $\dfrac{x^{2n} - 1}{x^2 - 1} : \dfrac{x^n - 1}{x - 1}$ will be a polynomial in x. We find

$$\frac{x^{2n} - 1}{x^2 - 1} : \frac{x^n - 1}{x - 1} = \frac{x^n + 1}{x + 1}.$$

For $x^n + 1$ to be divisible by $x + 1$ it is necessary and sufficient that $(-1)^n + 1 = 0$, i.e. that n is odd.

Thus, $1 + x^2 + \ldots + x^{2n-2}$ is divisible by $1 + x + x^2 + \ldots + x^{n-1}$ if n is odd.

22. 1° Put

$$f(x) = (\cos\varphi + x\sin\varphi)^n - \cos n\varphi - x\sin n\varphi.$$

But $x^2 + 1 = (x + i)(x - i)$ and $f(i) = (\cos\varphi + i\sin\varphi)^n - (\cos n\varphi + i\sin n\varphi) = 0$ (by de Moivre's formula). Likewise we make sure that $f(-i) = 0$, and our supposition is proved.

2° Resolve the polynomial $x^2 - 2\rho x\cos\varphi + \rho^2$ into factors linear in x. For this purpose find the roots of the quadratic equation

$$x^2 - 2\rho\, x\cos\varphi + \rho^2 = 0.$$

We get

$$x = \rho\cos\varphi \pm \sqrt{\rho^2\cos^2\varphi - \rho^2} = \rho(\cos\varphi \pm i\sin\varphi).$$

Let us denote

$$x^n \sin\varphi - \rho^{n-1}x\sin n\varphi + \rho^n\sin(n-1)\varphi = f(x).$$

We have to prove that

$$f[\rho(\cos\varphi \pm i\sin\varphi) = 0.$$

23. Suppose

$$x^4 + 1 = (x^2 + px + q)(x^2 + p'x + q') =$$
$$= x^4 + (p + p')x^3 + (q + q' + pp')x^2 + (pq' + qp')x + qq'.$$

For determining p, q, p' and q' we have four equations

$$p + p' = 0, \quad (1)$$
$$pp' + q + q' = 0, \quad (2)$$
$$pq' + qp' = 0, \quad (3)$$
$$qq' = 1. \quad (4)$$

From (1) and (3) we find $p' = -p$, $p(q' - q) = 0$.

1° Assume $p = 0$, $p' = 0$, $q + q' = 0$, $qq' = 1$, $q^2 = -1$, $q = \pm i$, $q' = \mp i$.

The corresponding factorization has the form

$$x^4 + 1 = (x^2 + i)(x^2 - i).$$

2° $q' = q$, $q^2 = 1$, $q = \pm 1$.

Suppose first $q' = q = 1$. Then $pp' = -2$, $p + p' = 0$, $p^2 = 2$, $p = \pm\sqrt{2}$, $p' = \mp\sqrt{2}$. The corresponding factorization is

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1).$$

Assume then

$$q = q' = -1, \quad p + p' = 0, \quad pp' = 2, \quad p = \pm\sqrt{2}i,$$
$$p' = \mp\sqrt{2}i.$$

The factorization will be

$$x^4 + 1 = (x^2 + \sqrt{2}ix - 1)(x^2 - \sqrt{2}ix - 1).$$

24. Put

$$\sqrt{a+bi} = x + yi,$$

whence

$$a + bi = x^2 - y^2 + 2xyi;$$

consequently,

$$x^2 - y^2 = a, \quad 2xy = b.$$

To find x and y it only remains to solve this system of two equations in two unknowns.

We have

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = a^2+b^2, \quad x^2+y^2 = \sqrt{a^2+b^2};$$

therefore

$$x^2 = a + \sqrt{a^2+b^2}, \qquad y^2 = -a + \sqrt{a^2+b^2},$$
$$x = \pm\sqrt{a+\sqrt{a^2+b^2}}, \quad y = \pm\sqrt{-a+\sqrt{a^2+b^2}},$$

the signs of the roots being related as $2xy = b$. And so, the following formula takes place

$$\sqrt{a+bi} = \pm\left(\sqrt{a+\sqrt{a^2+b^2}} + i\sqrt{-a+\sqrt{a^2+b^2}}\right)$$

if $b > 0$ (since then the signs of x and y must be the same), and

$$\sqrt{a+bi} = \pm\left(\sqrt{a+\sqrt{a^2+b^2}} - i\sqrt{-a+\sqrt{a^2+b^2}}\right)$$

if $b < 0$.

25. The roots of the given equation are determined by the formula

$$x_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} =$$
$$= \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k \quad (k = 0, 1, \ldots, n-1).$$

26. We have

$$s = \sum_{k=0}^{n-1} x_k^p = \sum_{k=0}^{n-1} \varepsilon^{kp},$$

where

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Thus

$$\sum_{k=0}^{n-1} x_k^p = 1 + \varepsilon^p + \varepsilon^{2p} + \ldots + \varepsilon^{(n-1)p}.$$

But

$$\varepsilon^p = \cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}.$$

It is obvious that $\varepsilon^p = 1$ if and only if p is divisible by n. In this case

$$s = n.$$

And if $\varepsilon^p \neq 1$, then $s = 1 + \varepsilon^p + \varepsilon^{2p} + \ldots + \varepsilon^{(n-1)p} = \frac{\varepsilon^{np} - 1}{\varepsilon^p - 1} = 0$, since $\varepsilon^{np} = 1$.

Thus

$$\sum_{k=0}^{n-1} x_k^p = n \text{ if } p \text{ is divisible by } n,$$

and

$$\sum_{k=0}^{n-1} x_k^p = 0 \text{ if } p \text{ is not divisible by } n.$$

27. We have

$$\sum_{k=0}^{n-1} |A_k|^2 = \sum_{k=0}^{n-1} A_k \bar{A}_k.$$

But

$$A_k\bar{A}_k=(x+y\varepsilon^k+z\varepsilon^{2k}+\ldots+w\varepsilon^{(n-1)k})\times$$
$$\times(\bar{x}+\bar{y}\varepsilon^{-k}+\bar{z}\varepsilon^{-2k}+\ldots+\bar{w}\varepsilon^{-(n-1)k})=$$
$$=x\bar{x}+y\bar{y}+\ldots+w\bar{w}+x(\bar{y}\varepsilon^{-k}+\bar{z}\varepsilon^{-2k}+\ldots+$$
$$+\bar{w}\varepsilon^{-(n-1)k})+y\varepsilon^k(\bar{x}+\bar{z}\varepsilon^{-2k}+\ldots+\bar{w}\varepsilon^{-(n-1)k})+$$
$$+z\varepsilon^{2k}(\bar{x}+\bar{y}\varepsilon^{-k}+\ldots+\bar{w}\varepsilon^{-(n-1)k})+$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+w\varepsilon^{(n-1)k}(\bar{x}+\bar{y}\varepsilon^{-k}+\ldots+\bar{u}\varepsilon^{-(n-2)k}).$$

Therefore

$$\sum_{k=0}^{n-1}A_k\bar{A}_k=n(|x|^2+|y|^2+\ldots+|w|^2)+$$
$$+x\sum_{k=0}^{n-1}(\bar{y}\varepsilon^{-k}+\bar{z}\varepsilon^{-2k}+\ldots+\bar{w}\varepsilon^{-(n-1)k})+$$
$$+y\sum_{k=0}^{n-1}(\bar{x}\varepsilon^{k}+\bar{z}\varepsilon^{-k}+\ldots+\bar{w}\varepsilon^{-(n-2)k})+\ldots+$$
$$+w\sum_{k=1}^{n-1}(\bar{x}\varepsilon^{(n-1)k}+\bar{y}\varepsilon^{(n-2)k}+\ldots+\bar{u}\varepsilon^{k}).$$

But $\sum_{k=0}^{n-1}\varepsilon^{lk}=0$ if l is not divisible by n (see Problem 26).

Therefore all the sums in the right member vanish and we get

$$|A_0|^2+|A_1|^2+\ldots+|A_{n-1}|^2=n\{|x|^2+|y|^2+\ldots+|w|^2\}.$$

28. 1° Denote the roots of index $2n$ from unity by x_s so that

$$x_s=\cos\frac{2s\pi}{n}+i\sin\frac{2s\pi}{n}\quad(s=1,\ 2,\ \ldots,\ 2n).$$

Therefore

$$x^{2n}-1=\prod_{s=1}^{2n}(x-x_s)=\prod_{s=1}^{n-1}(x-x_s)\prod_{s=n+1}^{2n-1}(x-x_s)\cdot(x^2-1),$$

since $x_n = -1$, $x_{2n} = 1$. But $x_{2n-s} = \bar{x}_s$, consequently,

$$x^{2n} - 1 = (x^2 - 1) \prod_{s=1}^{n-1} (x - x_s)(x - \bar{x}_s) =$$

$$= (x^2 - 1) \prod_{s=1}^{n-1} \left(x^2 - 2x \cos \frac{s\pi}{n} + 1 \right).$$

The rest of the cases are proved similarly.

29. 1° Rewrite the equality 1° of the preceding problem in the following way

$$x^{2n-2} + x^{2n-4} + \ldots + x^2 + 1 = \prod_{s=1}^{n-1} \left(x^2 - 2x \cos \frac{s\pi}{n} + 1 \right).$$

Put in this identity $x = 1$. We have

$$n = \prod_{s=1}^{n-1} \left(2 - 2 \cos \frac{s\pi}{n} \right) = \prod_{s=1}^{n-1} 4 \sin^2 \frac{s\pi}{n} =$$

$$= 2^{2(n-1)} \sin^2 \frac{\pi}{n} \cdot \sin^2 \frac{2\pi}{n} \ldots \sin^2 \frac{(n-1)\pi}{n}.$$

Hence

$$\sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n} = \frac{\sqrt{n}}{2^{n-1}}.$$

2° Solved analogously to 1°.

30. We have

$$x^n - 1 = (x - 1)(x - \alpha)(x - \beta)(x - \gamma) \ldots (x - \lambda).$$

Hence

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = (x - \alpha)(x - \beta) \ldots (x - \lambda).$$

Consequently

$$(1 - \alpha)(1 - \beta) \ldots (1 - \lambda) = n.$$

31. Set up an equation whose roots are

$$x_1 - 1, \quad x_2 - 1, \quad \ldots, x_n - 1.$$

This equation has the form

$$(x + 1)^n + (x + 1)^{n-1} + \ldots + (x + 1) + 1 = 0,$$

i.e.

$$\frac{(x+1)^{n+1}-1}{x+1-1}=\frac{(x+1)^{n+1}-1}{x}=0.$$

Then set up an equation with the roots

$$\frac{1}{x_1-1},\quad \frac{1}{x_2-1},\quad \ldots,\quad \frac{1}{x_n-1}.$$

It has the form

$$\frac{\left(\frac{1}{x}+1\right)^{n+1}-1}{\frac{1}{x}}=\frac{(1+x)^{n+1}-x^{n+1}}{x^n}=0.$$

Expanding the last expression in powers of x, we find

$$(n+1)\,x^n+\frac{(n+1)\,n}{1\cdot 2}\,x^{n-1}+\ldots=0$$

or

$$x^n+\frac{n}{2}\,x^{n-1}+\ldots .$$

The sum of the roots of this equation is equal to $-\frac{n}{2}$. Consequently

$$\frac{1}{x_1-1}+\frac{1}{x_2-1}+\ldots+\frac{1}{x_n-1}=-\frac{n}{2}.$$

32. Consider the equation (with t as an unknown)

$$\frac{x^2}{t}+\frac{y^2}{t-b^2}+\frac{z^2}{t-c^2}=1.$$

By virtue of the given equations this equation has three roots: μ^2, ν^2, ρ^2.
Expanding the last equation in powers of t, we get

$$t\,(t-b^2)\,(t-c^2)-x^2\,(t-b^2)\,(t-c^2)-$$
$$-\,y^2\,(t-c^2)\,t-z^2\,(t-b^2)\,t=0,$$
$$t^3+\alpha t^2+\ldots=0,$$

where $\alpha=-b^2-c^2-x^2-y^2-z^2$.

But as we know, the roots of this equation are μ^2, ν^2, ρ^2. Therefore, it must be

$$\mu^2+\nu^2+\rho^2=b^2+c^2+x^2+y^2+z^2.$$

Hence

$$x^2 + y^2 + z^2 = \mu^2 + \nu^2 + \rho^2 - b^2 - c^2.$$

33. Since $\cos\alpha + i\sin\alpha$ is the root of the given equation, we have

$$\sum_{k=0}^{n} p_k (\cos\alpha + i\sin\alpha)^{n-k} = 0 \quad (p_0 = 1)$$

or

$$(\cos\alpha + i\sin\alpha)^n \sum_{k=0}^{n} p_k (\cos\alpha + i\sin\alpha)^{-k} = 0.$$

But

$$(\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha,$$

therefore

$$\sum_{k=0}^{n} p_k (\cos\alpha - i\sin\alpha)^k = 0, \quad \sum_{k=0}^{n} p_k (\cos\alpha k - i\sin\alpha k) = 0.$$

Hence, indeed,

$$\sum_{k=0}^{n} p_k \sin k\alpha = p_1 \sin\alpha + p_2 \sin 2\alpha + \ldots + p_n \sin n\alpha = 0.$$

34. On the basis of the given data we have identically

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n =$$
$$= (x-a)(x-b) \ldots (x-k).$$

Substituting for x first i and then $-i$ and multiplying termwise, we get the required result.

35. Extracting the two given equations termwise, we find

$$(p - p')x + (q - q') = 0. \tag{1}$$

Multiplying the first equation by q' and the second by q and subtracting term by term, we have

$$\begin{aligned} x^3 (q' - q) + x (pq' - qp') &= 0 \\ x^2 (q' - q) + pq' - qp' &= 0. \end{aligned} \tag{2}$$

Eliminating then x from equations (1) and (2), we obtain the required result.

36. The roots of the equation

$$x^7 = 1$$

are

$$\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7} \quad (k=0,\ 1,\ 2,\ \ldots,\ 6).$$

Therefore, the roots of the equation

$$x^6+x^5+x^4+x^3+x^2+x+1=0 \qquad (*)$$

will be

$$x_k=\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7} \quad (k=1,\ 2,\ 3,\ 4,\ 5,\ 6).$$

Put

$$x+\frac{1}{x}=y,$$

then

$$x^2+\frac{1}{x^2}=y^2-2, \quad x^3+\frac{1}{x^3}=y^3-3y.$$

Equation (*) may be rewritten in the following way

$$\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0.$$

It is evident that

$$x_1=\bar{x}_6, \quad x_2=\bar{x}_5, \quad x_3=\bar{x}_4, \quad x_k+\frac{1}{x_k}=x_k+\bar{x}_k=2\cos\frac{2k\pi}{7}.$$

Hence, we may conclude that the quantities

$$2\cos\frac{2\pi}{7}, \quad 2\cos\frac{4\pi}{7}, \quad 2\cos\frac{8\pi}{7}$$

are the roots of the following equation

$$y^3+y^2-2y-1=0.$$

Let us set up an equation with the following roots

$$\sqrt[3]{2\cos\frac{2\pi}{7}}, \quad \sqrt[3]{2\cos\frac{4\pi}{7}}, \quad \sqrt[3]{2\cos\frac{8\pi}{7}}.$$

Let the roots of a certain cubic equation

$$x^3-ax^2+bx-c=0$$

be

$$\alpha,\ \beta,\ \gamma.$$

We then have

$$\alpha+\beta+\gamma=a, \quad \alpha\beta+\alpha\gamma+\beta\gamma=b, \quad \alpha\beta\gamma=c.$$

Let the equation, whose roots are the quantities $\sqrt[3]{\alpha}$, $\sqrt[3]{\beta}$, $\sqrt[3]{\gamma}$, be

$$x^3 - Ax^2 + Bx - C = 0.$$

Then

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=A,$$

$$\sqrt[3]{\alpha}\sqrt[3]{\beta}+\sqrt[3]{\alpha}\sqrt[3]{\gamma}+\sqrt[3]{\beta}\sqrt[3]{\gamma}=B, \quad \sqrt[3]{\alpha\beta\gamma}=C.$$

Let us make use of the following identity

$$(m+p+q)^3=m^3+p^3+q^3+$$
$$+3(m+p+q)(mp+mq+pq)-3mpq.$$

Putting here instead of m, p and q first $\sqrt[3]{\alpha}$, $\sqrt[3]{\beta}$, $\sqrt[3]{\gamma}$, and then $\sqrt[3]{\alpha\beta}$, $\sqrt[3]{\alpha\gamma}$, $\sqrt[3]{\beta\gamma}$, we find

$$A^3 = a + 3AB - 3C, \quad B^3 = b + 3BCA - 3C^2.$$

In our case we have $a = -1$, $b = -2$, $c = 1$, $C = 1$. Hence

$$A^3 = 3AB - 4, \quad B^3 = 3AB - 5.$$

Multiplying these equations and putting $AB = z$, we find

$$z^3 - 9z^2 + 27z - 20 = 0, \quad (z-3)^3 + 7 = 0,$$
$$z = 3 - \sqrt[3]{7}.$$

But

$$A^3=3z-4=5-3\sqrt[3]{7}, \quad A=\sqrt[3]{5-3\sqrt[3]{7}}.$$

Therefore, indeed,

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=$$
$$=\sqrt[3]{2\cos\frac{2\pi}{7}}+\sqrt[3]{2\cos\frac{4\pi}{7}}+\sqrt[3]{2\cos\frac{8\pi}{7}}=$$
$$=\sqrt[3]{5-3\sqrt[3]{7}}.$$

The second identity is proved in the same way.

37. Since by hypothesis $a + b + c = 0$, we may consider that a, b and c are the roots of the following equation

$$x^3 + px + q = 0,$$

where

$$p = ab + ac + bc, \quad q = -abc.$$

We have

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc),$$

i.e.

$$s_2 = -2p.$$

Putting in our equation in turn $x = a$, $x = b$, $x = c$, we get the following equalities

$$a^3+pa+q=0, \quad b^3+pb+q=0, \quad c^3+pc+q=0.$$

Adding them term by term, we find

$$s_3+ps_1+3q=0.$$

But since $s_1 = a+b+c = 0$, we have $s_3 = -3q$.

Multiplying both members of the original equation by x^k, putting then $x = a$, b and c, and adding, we find

$$s_{k+3} = -ps_{k+1} - qs_k.$$

Putting here $k = 1, 2, 3, 4$, we find

$$s_4 = 2p^2, \quad s_5 = 5pq, \quad s_6 = -2p^3+3q^2, \quad s_7 = -7p^2q.$$

Taking advantage of these relationships, we easily prove the first six formulas. The last one is also obtained readily.

38. We have

$$x-u = v-y, \quad x^2-u^2 = v^2-y^2.$$

The second equality may be rewritten as follows

$$(x-u)(x+u)-(v-y)(v+y)=0.$$

Since $x-u = v-y$, the last equality is rewritten as

$$(x-u)[x+u-(v+y)]=0,$$

wherefrom follows

1° $x-u=0$, $v-y=0$, $x=u$, $y=v$;

2° $(x+u)-(v+y)=0$, $(x-u)-(v-y)=0$, $x = v$, $y = u$.

Consequently, indeed,

$$x^n+y^n = u^n+v^n.$$

Let us go over to the second case. Suppose x, y, z are the roots of a cubic equation

$$\alpha^3 + p\alpha^2 + q\alpha + r = 0.$$

Prove that u, v and t are the roots of the same equation. We have

$$x + y + z = -p, \quad xy + xz + yz = q, \quad xyz = -r.$$

Hence, to prove that u, v, and t are the roots of the same equation (whose roots are x, y and z) it is sufficient to prove that

$$u + v + t = x + y + z, \quad uv + ut + vt = xy + xz + yz, \quad uvt = xyz.$$

The first of these equalities is true by hypothesis. The second one follows immediately from the identity

$$2(xy + xz + yz) = (x + y + z)^2 - (x^2 + y^2 + z^2)$$

and from the condition

$$x^2 + y^2 + z^2 = u^2 + v^2 + t^2.$$

Likewise, the third equality follows from the identity

$$3xyz = x^3 + y^3 + z^3 + 3(x + y + z) \times (xy + xz + yz) - (x + y + z)^3$$

and from the condition

$$x^3 + y^3 + z^3 = u^3 + v^3 + t^3.$$

Thus, u, v, t as well as x, y, z are the roots of the same third-degree equation. Therefore, one of the six possibilities takes place

u	v	t
x	y	z
y	x	z
x	z	y
y	z	x
z	x	y
z	y	x

It is obvious that in all cases we have

$$x^n + y^n + z^n = u^n + v^n + t^n.$$

39. Squaring the first trinomial, we get

$$A^2 = (x_1^2 + 2x_2x_3) + (x_3^2 + 2x_1x_2)\,\varepsilon + (x_2^2 + 2x_1x_3)\,\varepsilon^2.$$

Then

$$A^3 = (x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3) + (3x_1^2x_2 + 3x_2^2x_1 + 3x_3^2x_3)\,\varepsilon + \\ + (3x_1^2x_3 + 3x_2^2x_1 + 3x_3^2x_2)\,\varepsilon^2.$$

Put

$$\alpha = x_1^2x_2 + x_2^2x_3 + x_3^2x_1, \quad \beta = x_1x_2^2 + x_2x_3^2 + x_3x_1^2.$$

Now

$$x_1^3 + x_2^3 + x_3^3 = -(px_1 + q) - (px_2 + q) - (px_3 + q) = -3q,$$

since

$$x_1 + x_2 + x_3 = 0.$$

Furthermore

$$x_1x_2x_3 = -q,$$

therefore

$$A^3 = -9q + 3\alpha\varepsilon + 3\beta\varepsilon^2.$$

Substituting x_2 and x_3, we also find

$$B^3 = -9q + 3\alpha\varepsilon^2 + 3\beta\varepsilon.$$

Hence

$$A^3 + B^3 = -18q - 3\alpha - 3\beta = -27q,$$

since

$$\alpha + \beta = x_1x_2(x_1 + x_2) + x_2x_3(x_2 + x_3) + \\ + x_3x_1(x_3 + x_1) = -3x_1x_2x_3 = 3q.$$

Likewise we get

$$A^3 \cdot B^3 = -27p^3.$$

It should be taken into consideration that

$$\alpha\beta = 3x_1^2x_2^2x_3^2 + (x_1^3x_2^3 + x_1^3x_3^3 + x_2^3x_3^3) + x_1^4x_2x_3 + x_2^4x_1x_3 + \\ + x_3^4x_2x_1 = 3q^2 + x_1^3x_2^3x_3^3\left(\frac{1}{x_1^3} + \frac{1}{x_2^3} + \frac{1}{x_3^3}\right) + \\ + x_1x_2x_3(x_1^3 + x_2^3 + x_3^3),$$

and

$$\frac{1}{x_1^3} + \frac{1}{x_2^3} + \frac{1}{x_3^3} = -\frac{3}{q} - \frac{p^3}{q^3}.$$

40. Put

$$a+b=c+d=p.$$

We have

$$(x^2+px+ab)(x^2+px+cd)=m$$

or

$$\left[\left(x+\frac{p}{2}\right)^2+ab-\frac{p^2}{4}\right]\left[\left(x+\frac{p}{2}\right)^2+cd-\frac{p^2}{4}\right]=m.$$

Let

$$\left(x+\frac{p}{2}\right)^2=y.$$

Then the equation takes the form

$$\left(y+ab-\frac{p^2}{4}\right)\left(y+cd-\frac{p^2}{4}\right)=m,$$

i.e.

$$y^2+\left(ab+cd-\frac{p^2}{2}\right)y+\left(ab-\frac{p^2}{4}\right)\left(cd-\frac{p^2}{4}\right)-m=0.$$

It only remains to solve this quadratic equation.

41. Make the following substitution

$$x=y-\frac{a+b}{2},$$

then

$$x+a=y+\frac{a-b}{2}, \quad x+b=y-\frac{a-b}{2}.$$

The equation takes the form

$$\left(y+\frac{a-b}{2}\right)^4+\left(y-\frac{a-b}{2}\right)^4=c.$$

But

$$\left(y+\frac{a-b}{2}\right)^4=y^4+4y^3\frac{a-b}{2}+6y^2\left(\frac{a-b}{2}\right)^2+$$
$$+4y\left(\frac{a-b}{2}\right)^3+\left(\frac{a-b}{2}\right)^4.$$

Therefore the equation takes the form

$$y^4+6\left(\frac{a-b}{2}\right)^2y^2+\left(\frac{a-b}{2}\right)^4=\frac{c}{2}.$$

Thus, the problem is reduced to solving a biquadratic equation.

42. Put for brevity

$$a + b + c = p$$

and make the substitution

$$x + p = y.$$

We have

$$(y - a)(y - b)(y - c)p - abc(y - p) = 0.$$

Hence

$$p\{y^3 - (a + b + c)y^2 + (ab + ac + bc)y\} - abcy = 0$$

or

$$y\{(a + b + c)y^2 - (a + b + c)^2 y + (ab + ac + bc)(a + b + c) - abc\} = 0.$$

And so, we find three values for y: one of them is zero, the other two are obtained as the roots of a quadratic equation. Then it is easy to find the corresponding values of x.

43. Rewrite the equation in the following way

$$(x + a)^3 - 3bc(x + a) + b^3 + c^3 = 0.$$

Put $x + a = y$. The equation takes the form

$$y^3 - 3bcy + b^3 + c^3 = 0.$$

But it is known (Problem 20, Sec. 1) that

$$y^3 + b^3 + c^3 - 3bcy = (y + b + c)(y^2 + b^2 + c^2 - yb - yc - bc).$$

Consequently, one of the roots of the last equation will be $-b - c$, the other two are found by solving the quadratic equation. Then we find the corresponding values of x.

44. The equation contains five coefficients: a, b, c, d and e, and there exist two relationships among them. Thus, three coefficients remain arbitrary. Let us express all the coefficients in terms of any three.

We have

$$a = c + d, \quad e = b + c.$$

The equation takes the form

$$(c + d)x^4 + bx^3 + cx^2 + dx + (b + c) = 0,$$

$$c(x^4 + x^2 + 1) + dx(x^3 + 1) + b(x^3 + 1) = 0.$$

But

$$x^3 + 1 = (x + 1)(x^2 - x + 1),$$

$$x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2 = (x^2 + 1)^2 - x^2 =$$
$$= (x^2 + x + 1)(x^2 - x + 1).$$

The equation is now rewritten as

$$(x^2 - x + 1)\{c(x^2 + x + 1) + dx(x + 1) +$$
$$+ b(x + 1)\} = 0.$$

Equating the first factor to zero, we find

$$x = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The remaining two roots are found by solving the second quadratic equation.

45. We have the following formula

$$(a + b + x)^3 = a^3 + b^3 + x^3 + 3a^2(b + x) +$$
$$+ 3b^2(a + x) + 3x^2(a + b) + 6\,abx.$$

Using this formula, reduce our equation to the form

$$x^3 - (a + b)x^2 - (a - b)^2 x + (a - b)^2(a + b) = 0.$$

Hence

$$x^2(x - a - b) - (a - b)^2(x - a - b) = 0,$$
$$(x - a - b)[x^2 - (a - b)^2] = 0,$$
$$(x - a - b)(x + a - b)(x - a + b) = 0.$$

Thus, the given equation has three roots:

$$x = a + b, \quad x = a - b, \quad x = b - a.$$

46. Rewrite the equation as follows

$$x^2 + \frac{a^2x^2}{(a+x)^2} - \frac{2ax^2}{a+x} = m^2 - \frac{2ax^2}{a+x}.$$

Consequently

$$\left(x - \frac{ax}{a+x}\right)^2 = m^2 - \frac{2ax^2}{a+x}.$$

Hence

$$\frac{x^4}{(a+x)^2} = m^2 - \frac{2ax^2}{a+x}.$$

Put $\frac{x^2}{a+x}=y$. Then the equation takes the form

$$y^2+2ay-m^2=0,$$

wherefrom we find y and then x. For y we find the following values

$$y=-a\pm\sqrt{a^2+m^2}. \qquad (1)$$

The corresponding values of x are determined by the formula

$$x=\frac{y}{2}\pm\sqrt{\frac{y^2}{4}+ay}. \qquad (2)$$

Let us take the plus sign in formula (1). In this case the value of y will exceed zero. Computing, by formula (2), the corresponding values of x, we make sure that x has two values: one positive, the other negative. And so, our equation always has at least two real roots, positive and negative.

Consider the case when the minus sign is taken in formula (1). Now the value of y is negative, and for x to be real it is necessary and sufficient that $y^2+4ay\geqslant 0$. And, consequently, it must be

$$y+4a\leqslant 0,$$

i.e.

$$-a-\sqrt{a^2+m^2}+4a\leqslant 0,$$

$$m^2\geqslant 8a^2.$$

With this condition satisfied, all the four roots will be real. Since $ay<0$, we have

$$\left|\sqrt{\frac{y^2}{4}+ay}\right|<\left|\frac{y}{2}\right|$$

and, consequently, both real roots, found from formula (1) taken with the minus sign, will be negative. Thus, if all the four roots are real, then one of them is positive, the remaining being negative.

47. Put for brevity

$$\frac{5x^4+10x^2+1}{x^4+10x^2+5}=f(x).$$

Then the equation takes the form

$$f(x)\cdot f(a)=ax.$$

Further, we have

$$x-f(x)=\frac{(x-1)^5}{x^4+10x^2+5}, \qquad x+f(x)=\frac{(x+1)^5}{x^4+10x^2+5}.$$

Dividing the first equation by the second one, we find

$$\frac{x-f(x)}{x+f(x)}=\left(\frac{x-1}{x+1}\right)^5. \tag{*}$$

Put

$$\frac{x-1}{x+1}=y, \qquad \frac{a-1}{a+1}=b.$$

From the equation (*) we get

$$x-f(x)=y^5x+y^5f(x), \qquad x(1-y^5)=f(x)(1+y^5),$$

$$\frac{f(x)}{x}=\frac{1-y^5}{1+y^5}.$$

Likewise we have

$$\frac{f(a)}{a}=\frac{1-b^5}{1+b^5}.$$

Now our equation can be rewritten in the following way

$$\frac{1-y^5}{1+y^5}=\frac{1+b^5}{1-b^5},$$

whence

$$y^5=-b^5.$$

The last equation has five roots, namely

$$y_k=-b\varepsilon^k \quad (k=0,\ 1,\ 2,\ 3,\ 4);$$

$$\varepsilon=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}.$$

But

$$x=\frac{1+y}{1-y},$$

consequently

$$x_k=\frac{1+y_k}{1-y_k}=\frac{1-b\varepsilon^k}{1+b\varepsilon^k}=\frac{(a+1)-(a-1)\varepsilon^k}{(a+1)+(a-1)\varepsilon^k}.$$

Further

$$x_k=\frac{(a+1)\varepsilon^{-\frac{k}{2}}-(a-1)\varepsilon^{\frac{k}{2}}}{(a+1)\varepsilon^{-\frac{k}{2}}+(a-1)\varepsilon^{\frac{k}{2}}}=$$

$$=\frac{a\left(\varepsilon^{-\frac{k}{2}}-\varepsilon^{\frac{k}{2}}\right)+\varepsilon^{-\frac{k}{2}}+\varepsilon^{\frac{k}{2}}}{a\left(\varepsilon^{-\frac{k}{2}}+\varepsilon^{\frac{k}{2}}\right)+\varepsilon^{-\frac{k}{2}}-\varepsilon^{\frac{k}{2}}}=\frac{\cos\frac{\pi k}{5}-ia\sin\frac{\pi k}{5}}{a\cos\frac{\pi k}{5}-i\sin\frac{\pi k}{5}}.$$

In particular, at $k=0$ the solution is

$$x_0=\frac{1}{a}.$$

48. Transform the left member of the equation. Denote the sum on the left by S_m. Then

$$S_1=1+\frac{a_1}{x-a_1}+\frac{a_2x}{(x-a_1)(x-a_2)}=\frac{x^2}{(x-a_1)(x-a_2)}.$$

Prove that

$$S_m=\frac{x^{2m}}{(x-a_1)(x-a_2)\dots(x-a_{2m})}.$$

Suppose this equality is true at $m=n$, and prove that it will be true also at $m=n+1$. We have

$$S_{n+1}=\frac{x^{2n}}{(x-a_1)\dots(x-a_{2n})}+\frac{a_{2n+1}x^{2n}}{(x-a_1)\dots(x-a_{2n})(x-a_{2n+1})}+$$
$$+\frac{a_{2n+2}x^{2n+1}}{(x-a_1)\dots(x-a_{2n+2})}.$$

Reducing the right member to a common denominator and accomplishing all the necessary transformations, we get

$$S_{n+1}=\frac{x^{2n+2}}{(x-a_1)\dots(x-a_{2n+2})}.$$

Now our equation takes the form

$$\frac{x^{2m}-2px^m+p^2}{(x-a_1)\dots(x-a_{2m})}=0$$

or

$$(x^m-p)(x^m-p)=0.$$

The equation has m double roots.

49. 1° We have $x_1 + x_2 + x_3 = -p$, $x_1x_2 + x_1x_3 + x_2x_3 = q$, $x_1x_2x_3 = -r$.

From the second equality we get

$$x_1x_2 + x_1x_3 + x_1^2 = x_1(x_1 + x_2 + x_3) = q,$$

whence

$$x_1 = -\frac{q}{p}.$$

Using the first equality, we find

$$x_2 + x_3 = \frac{q - p^2}{p}.$$

From the third equality we have

$$x_2x_3 = \frac{rp}{q}.$$

It only remains to set up a quadratic equation satisfied by x_2 and x_3.

2° Solved analogously to the preceding one.

50. 1° Using the identity of Problem 4 of this section, we can rewrite our system in the following way

$$(y + z + a)(y + z\varepsilon + a\varepsilon^2)(y + z\varepsilon^2 + a\varepsilon) = 0$$

$$(z + x + b)(z + x\varepsilon + b\varepsilon^2)(z + x\varepsilon^2 + b\varepsilon) = 0$$

$$(x + y + c)(x + y\varepsilon + c\varepsilon^2)(x + y\varepsilon^2 + c\varepsilon) = 0.$$

To find all the solutions of the given system it is necessary to consider all possible (27) combinations. Thus, we get 27 systems, each containing three equations linear in the unknowns x, y, and z.

If each of these systems is designated by a three-digit number in which the place occupied by a certain digit corresponds to the number of the equation and the digit itself to the number of the factor in this equation, then the 27 systems will be written as

111, 112, 113, 121, 122, 123, 131, 132, 133,

211, 212, 213, 221, 222, 223, 231, 232, 233,

311, 312, 313, 321, 322, 323, 331, 332, 333.

Let us explain, for example, system 213: taken from the first equation is the second factor, from the second—

the first factor and from the third—the third factor. Thus, system 213 will have the following form

$$y + z\varepsilon + a\varepsilon^2 = 0, \quad z + x + b = 0, \quad x + y\varepsilon^2 + c\varepsilon = 0.$$

Let us decipher some more systems

$$y+z+a=0, \quad z+x+b=0, \quad x+y+c=0; \quad (111)$$
$$y+z\varepsilon+a\varepsilon^2=0, \quad z+x\varepsilon^2+b\varepsilon=0, \quad x+y\varepsilon+c\varepsilon^2=0; \quad (232)$$
$$y+z\varepsilon^2+a\varepsilon=0, \quad z+x\varepsilon^2+b\varepsilon=0, \quad x+y\varepsilon^2+c\varepsilon=0; \quad (333)$$
$$y+z+a=0, \quad z+x\varepsilon+b\varepsilon^2=0, \quad x+y\varepsilon+c\varepsilon^2=0 \quad (122)$$

and so on.

2° We have

$$x^4 = xyzu + a, \quad y^4 = xyzu + b, \quad z^4 = xyzu + c,$$
$$u^4 = xyzu + d.$$

Multiplying these equations and putting $xyzu = t$, we find

$$t^4 = (t + a)(t + b)(t + c)(t + d).$$

Thus, for determining t, we have the following equation

$$(a + b + c + d)\, t^3 + (ab + ac + \ldots)\, t^2 +$$
$$+ (abc + acd + \ldots)\, t + abcd = 0.$$

However,

$$a + b + c + d = 0,$$

therefore, for finding t we get a quadratic equation. Knowing t, we easily obtain x, y, z and u.

51. We have

$$1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^n = \frac{(1+x)^{n+1} - 1}{(1+x) - 1} =$$
$$= \frac{1}{x}\left\{\sum_{k=0}^{n+1} C_{n+1}^k x^k - 1\right\} = \sum_{k=1}^{n+1} C_{n+1}^k x^{k-1}.$$

Wherefrom follows that the term containing x^k will be

$$C_{n+1}^{k+1} x^k.$$

52. We have

$$(x+1)^n = 1 + C_n^1 x + C_n^2 x^2 + \ldots + C_n^{s-1} x^{s-1} + C_n^s x^s + \ldots + x^n.$$

Since this polynomial is multiplied by the second-degree trinomial

$$(s-2)x^2+nx-s,$$

it is clear that the coefficient of x^s in the product will be equal to

$$(s-2)C_n^{s-2}+nC_n^{s-1}-sC_n^s.$$

Carrying out all the necessary transformations, we see that the last expression is equal to

$$nC_n^{s-2}.$$

53. Put $x=1+\alpha$, where $\alpha>0$ (since $x>1$). Then we have

$$px^q-qx^p-p+q=p(1+\alpha)^q-q(1+\alpha)^p-p+q=$$
$$=p\left\{1+q\alpha+\frac{q(q-1)}{1\cdot 2}\alpha^2+\dots\right\}-$$
$$-q\left\{1+p\alpha+\frac{p(p-1)}{1\cdot 2}\alpha^2+\dots\right\}-p+q=$$
$$=(pC_q^2-qC_p^2)\alpha^2+(pC_q^3-qC_p^3)\alpha^3+\dots .$$

Since $q>p$, we can prove that all the terms of the above expansion are positive [the coefficient of α^k (if $k>p$) will be equal to pC_q^k]. Thus, to prove the validity of our assertion, it is sufficient to prove that

$$\Delta=pC_q^k-qC_p^k>0$$

if $q>p$ and $k\leqslant p$.

We have

$$\Delta=p\frac{q(q-1)\dots(q-k+1)}{1\cdot 2\cdot 3\dots k}-q\frac{p(p-1)\dots(p-k+1)}{1\cdot 2\cdot 3\dots k}=$$
$$=\frac{pq}{k!}\{(q-1)(q-2)\dots(q-k+1)-(p-1)(p-2)\dots\times$$
$$\times(p-k+1)\}>0,$$

since

$$q-1>p-1,\quad q-2>p-2,\ \dots .$$

54. Let the greatest term be

$$T_k=C_n^kx^{n-k}a^k.$$

This term must not be less than the two neighbouring terms T_{k-1} and T_{k+1}. Thus, there exist the following inequalities

$$T_k \geqslant T_{k-1}, \quad T_k \geqslant T_{k+1}.$$

Whence

$$\frac{k}{n-k+1}\cdot\frac{x}{a}\leqslant 1, \quad \frac{n-k}{k+1}\cdot\frac{a}{x}\leqslant 1.$$

The first of them yields

$$k\leqslant\frac{(n+1)\,a}{x+a}.$$

From the second one we get

$$k\geqslant\frac{(n+1)\,a}{x+a}-1.$$

First assume the $\frac{(n+1)\,a}{x+a}$ is a whole number. Then $\frac{(n+1)\,a}{x+a}-1$ is also a whole number, and since k is a whole number satisfying the inequalities

$$\frac{(n+1)\,a}{x+a}-1\leqslant k\leqslant\frac{(n+1)\,a}{x+a},$$

it can attain two values

$$k=\frac{(n+1)\,a}{x+a}, \quad k=\frac{(n+1)\,a}{x+a}-1.$$

In this case there are two adjacent terms which are equal to each other but exceed all the rest of the terms. Now consider the case when $\frac{(n+1)a}{x+a}$ is not a whole number. We then have

$$\frac{(n+1)\,a}{x+a}=\left[\frac{(n+1)\,a}{x+a}\right]+\theta,$$

where $0<\theta<1$ (for the symbol [ ] see Problem 35, Sec. 1). In this case the inequalities take the form

$$k\leqslant\left[\frac{(n+1)\,a}{x+a}\right]+\theta, \quad k\geqslant\left[\frac{(n+1)\,a}{x+a}\right]-(1-\theta).$$

It is apparent that in this case there exists only one value of k at which our inequalities are satisfied, namely

$$k=\left[\frac{(n+1)\,a}{x+a}\right].$$

And so, when $\frac{(n+1)a}{x+a}$ is not a whole number, there exists only one greatest term T_k.

55. Let i and m be positive integers. We have

$$(x+1)^m - x^m = mx^{m-1} + \frac{m(m-1)}{1\cdot 2}x^{m-2} + \ldots + mx + 1.$$

Replacing here x by $x+1$, we get

$$(x+2)^m - (x+1)^m =$$
$$= m(x+1)^{m-1} + \frac{m(m-1)}{1\cdot 2}(x+1)^{m-2} + \ldots + m(x+1) + 1.$$

Subtracting the preceding equality from the last one, we find

$$(x+2)^m - 2(x+1)^m + x^m = m(m-1)x^{m-2} + p_1x^{m-3} + \ldots.$$

Analogously we obtain

$$(x+3)^m - 3(x+2)^m + 3(x+1)^m - x^m =$$
$$= m(m-1)(m-2)x^{m-3} + p_2x^{m-4} + \ldots.$$

Using the method of mathematical induction, we can prove the following general identity

$$(x+i)^m - \frac{i}{1}(x+i-1)^m + \frac{i(i-1)}{1\cdot 2}(x+i-2)^m + \ldots +$$
$$+(-1)^i x^m = m(m-1)\ldots(m-i+1)x^{m-i} + px^{m-i-1} + \ldots,$$

wherefrom it is easy to obtain that at $i=m$

$$(x+m)^m - \frac{m}{1}(x+m-1)^m + \ldots + (-1)^m x^m = m!.$$

If $i > m$, we get

$$(x+i)^m - \frac{i}{1}(x+i-1)^m +$$
$$+ \frac{i(i-1)}{1\cdot 2}(x+i-2)^m + \ldots + (-1)^i x^m = 0.$$

Putting in the last equalities $x=0$, we find the required identities.

56. We have

$$(x+ai)^n = x^n + C_n^1 x^{n-1} ai + C_n^2 x^{n-2}a^2i^2 + C_n^3 x^{n-3}a^3i^3 + \ldots =$$
$$= \{x^n - C_n^2 x^{n-2}a^2 + C_n^4 x^{n-4}a^4 - \ldots\} +$$
$$+ i\{C_n^1 x^{n-1}a - C_n^3 x^{n-3}a^3 + \ldots\}.$$

Going over to the conjugate quantities, we get

$$(x-ai)^n = \{x^n - C_n^2 x^{n-2}a^2 + C_n^4 x^{n-4}a^4 - \ldots\} - \\ - i\{C_n^1 x^{n-1}a - C_n^3 x^{n-3}a^3 + \ldots\}.$$

Multiplying these equalities term by term, we find the required result.

57. 1° We can write our product in the following way

$$\sum_{s=0}^{n} x^s \sum_{t=0}^{n} x^t = \sum_{l=0}^{2n} A_l x^l,$$

wherefrom it follows that

$$A_l = \sum_{\substack{s+t=l \\ 0\leqslant s\leqslant n \\ 0\leqslant t\leqslant n}} 1.$$

First assume $l \leqslant n$. Then s can attain the values $s = 0$, 1, 2, . . ., l and, consequently,

$$A_l = l + 1$$

if $l \leqslant n$.

If $n < l \leqslant 2n$, then we put

$$l = n + l',$$

where $1 \leqslant l' \leqslant n$, $l' = l - n$.

In this case s can take only the following values

$$s = l', \ l' + 1, \ \ldots, \ n.$$

The total number of values will be

$$n - (l' - 1) = n - (l - n - 1) = 2n - l + 1.$$

And so,

$$A_l = 2n + 1 - l \text{ if } n < l \leqslant 2n.$$

It is easily seen that $A_{n-k} = A_{n+k} = n - k + 1$.

Indeed, expanding the product, we get immediately

$$(1+x+x^2+\ldots+x^n)(1+x+x^2+\ldots+x^n) = \\ = 1+2x+3x^2+\ldots+nx^{n-1}+ \\ +(n+1)x^n+nx^{n+1}+\ldots+2x^{2n-1}+x^{2n}.$$

2° In this case we have

$$\sum_{s=0}^{n} (-1)^s x^s \sum_{t=0}^{n} x^t = \sum_{l=0}^{2n} A_l x^l.$$

Hence

$$A_l = \sum_{\substack{l=s+t \\ 0 \leqslant s \leqslant n \\ 0 \leqslant t \leqslant n}} (-1)^s.$$

Considering again separately the cases when $l \leqslant n$ and $l > n$, we arrive at the following conclusion

if $l \leqslant n$, then $A_l = \frac{1+(-1)^l}{2}$,

if $l > n$, then $A_l = 0$ when l is odd and

$A_l = (-1)^n$ when l is even.

Thus, $A_l = 0$ for any odd l, i.e. the product contains only even powers of x, and if n is even, then all the coefficients (of even powers) are equal to $+1$; if n is odd, then half of them is equal to $+1$, the other half to -1

$$A_0 = A_2 = \ldots = A_{n-1} = +1,$$

$$A_{n+1} = A_{n+3} = \ldots = A_{2n} = -1.$$

3° We have

$$\sum_{k=0}^{n} (k+1)\,x^k \sum_{s=0}^{n} (s+1)\,x^s = \sum_{l=0}^{2n} A_l x^l.$$

Hence

$$A_l = \sum_{\substack{k+s=l \\ 0 \leqslant k \leqslant n \\ 0 \leqslant s \leqslant n}} (k+1)(s+1) = \sum_{\substack{k+s=l \\ 0 \leqslant k \leqslant n \\ 0 \leqslant s \leqslant n}} (ks+l+1).$$

Let us first assume that $l \leqslant n$, then k can take on only the following values: 0, 1, 2, . . ., l, the corresponding values of s being l, $l-1$, . . ., 0.

Therefore

$$A_l = \sum_{k=0}^{l} [k(l-k)+l+1] -$$

$$= l \sum_{k=0}^{l} k - \sum_{k=0}^{l} k^2 + (l+1)^2 = \frac{(l+1)(l+2)(l+3)}{6},$$

taking as known that

$$1^2 + 2^2 + \ldots + l^2 = \frac{l(l+1)(2l+1)}{6}$$

(see Problem 25, Sec. 7).

Then assume $n < l \leqslant 2n$ and put $l = n + l'$, where $1 \leqslant l' \leqslant n$. Then k can attain only the following values

$$l',\ l' + 1,\ \ldots,\ n$$

and, consequently,

$$A_l = \sum_{k+s=l} (ks + l + 1) = \sum_{k=l-s} [k(l-k) + l + 1] =$$

$$= l \sum_{k=l-n}^{n} k - \sum_{k=l-n}^{n} k^2 + (l+1)(2n-l+1) =$$

$$= \frac{(2n-l+1)(l^2+2l+2)}{2} + \frac{(l-n-1)(l-n)(2l-2n-1)}{6} - \frac{n(n+1)(2n+1)}{6}.$$

4° Solved as the preceding case.

58. 1° We have

$$1 + C_n^1 + C_n^2 + C_n^3 + \ldots + C_n^{n-1} + C_n^n = (1+1)^n = 2^n,$$

$$1 - C_n^1 + C_n^2 - C_n^3 + \ldots + (-1)^n C_n^n = (1-1)^n = 0.$$

Adding the two equalities and then subtracting, we get the required identity.

2° as well as 3° is reduced to 1° if we take into account that

$$C_{2n}^k = C_{2n}^{2n-k}.$$

59. Consider the identity

$$(1+x)^n = C_n^0 + C_n^1 x + C_n^2 x^2 + C_n^3 x^3 + \ldots + C_n^{n-1} x^{n-1} + C_n^n x^n.$$

Putting in this identity in succession $x = 1,\ \varepsilon,\ \varepsilon^2$, where $\varepsilon^2 + \varepsilon + 1 = 0$, we get

$$2^n = C_n^0 + C_n^1 + C_n^2 + C_n^3 + \ldots$$

$$(1+\varepsilon)^n = C_n^0 + C_n^1 \varepsilon + C_n^2 \varepsilon^2 + C_n^3 \varepsilon^3 + \ldots$$

$$(1+\varepsilon^2)^n = C_n^0 + C_n^1 \varepsilon^2 + C_n^2 \varepsilon^4 + C_n^3 \varepsilon^6 + \ldots.$$

But $1 + \varepsilon^k + \varepsilon^{2k} = 0$ if k is not divisible by 3 and $1 + \varepsilon^k + \varepsilon^{2k} = 3$ if k is divisible by 3.

Consequently,

$$2^n + (1+\varepsilon)^n + (1+\varepsilon^2)^n = 3\{C_n^0 + C_n^3 + C_n^6 + \ldots\}.$$

Since for ε we can take the value

$$\varepsilon=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},$$

we have

$$1+\varepsilon=-\varepsilon^2=-\cos\frac{4\pi}{3}-i\sin\frac{4\pi}{3}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$1+\varepsilon^2=-\varepsilon=-\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}=\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}.$$

Therefore

$$2^n+(1+\varepsilon)^n+(1+\varepsilon^2)^n=2^n+2\cos\frac{n\pi}{3}.$$

Hence, we obtain

$$C_n^0+C_n^3+C_n^6+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right),$$

the other two equalities are obtained similarly by considering the sums

$$2^n+\varepsilon(1+\varepsilon)^n+\varepsilon^2(1+\varepsilon^2)^n,$$
$$2^n+\varepsilon^2(1+\varepsilon)^n+\varepsilon(1+\varepsilon^2)^n.$$

60. The solution is analogous to that of the preceding problem. Consider $(1+i)^n$.

61. Since $C_k^2=\frac{k(k-1)}{1\cdot 2}=\frac{k^2}{2}-\frac{k}{2}$, we get

$$2C_k^2=k^2-k.$$

Consequently,

$$2\sum_{k=2}^{n}C_k^2=\sum_{k=2}^{n}k^2-\sum_{k=2}^{n}k,$$

wherefrom our identity is obtained.

62. Let $a_1=C_n^k$, $a_2=C_n^{k+1}$, $a_3=C_n^{k+2}$, $a_4=C_n^{k+3}$. Then

$$\frac{a_2}{a_1}=\frac{n-k}{k+1},\quad \frac{a_4}{a_3}=\frac{n-k-2}{k+3},\quad \frac{a_3}{a_2}=\frac{n-k-1}{k+2}.$$

It only remains to prove that

$$\frac{1}{1+\frac{a_2}{a_1}}+\frac{1}{1+\frac{a_4}{a_3}}=\frac{2}{1+\frac{a_3}{a_2}}.$$

63. If we rewrite the equality in the form

$$\frac{n!}{1!\,(n-1)!}+\frac{n!}{3!\,(n-3)!}+\frac{n!}{5!\,(n-5)!}+\ldots+\frac{n!}{(n-1)!\,1!}=2^{n-1},$$

then the problem is reduced to proving the following relationship (see Problem 58)

$$C_n^1+C_n^3+\ldots+C_n^{n-1}=2^{n-1}.$$

64. Consider the equality

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^n=$$

$$=\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}. \qquad (*)$$

Further

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n=\frac{(-1)^n}{2^n}(1-i\sqrt{3})^n=$$

$$=\frac{(-1)^n}{2^n}\{1+C_n^1(-i\sqrt{3})+$$

$$+C_n^2(-i\sqrt{3})^2+C_n^3(-i\sqrt{3})^3+\ldots\}=$$

$$=\frac{(-1)^n}{2^n}\{1-3C_n^2+\ldots-$$

$$-i\sqrt{3}(C_n^1-3C_n^3+3^2C_n^5-3^3C_n^7+\ldots)\}.$$

Equating the coefficients of i in both members of the equality (*), we get

$$-\sqrt{3}(C_n^1-3C_n^3+3^2C_n^5-3^3C_n^7+\ldots)=(-1)^n 2^n\sin\frac{2n\pi}{3}.$$

Hence

$$s=C_n^1-3C_n^3+3^2C_n^5-3^3C_n^7+\ldots=(-1)^{n+1}\frac{2^n}{\sqrt{3}}\sin\frac{2n\pi}{3},$$

wherefrom we easily obtain

$$s=0 \quad \text{if } n\equiv 0 \pmod 3,$$

$$s=2^{n-1} \quad \text{if } n\equiv 1 \text{ or } 2 \pmod 6,$$

$$s=-2^{n-1} \quad \text{if } n\equiv 4 \text{ or } 5 \pmod 6.$$

65. Consider the expression

$$(1+i)^n.$$

We have

$$(1+i)^n = 1 + C_n^1 i + C_n^2 i^2 + C_n^3 i^3 + \dots .$$

Hence

$$(1+i)^n = (1 - C_n^2 + C_n^4 - C_n^6 + \dots) + i(C_n^1 - C_n^3 + C_n^5 - \dots).$$

But

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Therefore

$$\sigma = 1 - C_n^2 + C_n^4 - C_n^6 + \dots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4},$$

$$\sigma' = C_n^1 - C_n^3 + C_n^5 - C_n^7 + \dots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

Hence, if $n \equiv 0 \pmod 4$, i.e. $n = 4m$, then

$$\sigma = (-1)^m 2^{2m}, \quad \sigma' = 0.$$

If $n \equiv 1 \pmod 4$, i.e. $n = 4m+1$, then

$$\sigma = \sigma' = (-1)^m 2^{2m}.$$

If $n \equiv 3 \pmod 4$, i.e. $n = 4m+3$, then

$$\sigma = (-1)^{m+1} 2^{2m+1}, \quad \sigma' = (-1)^m 2^{2m+1}.$$

Finally, if $n \equiv 2 \pmod 4$, i.e. $n = 4m+2$, then

$$\sigma = 0, \quad \sigma' = (-1)^m 2^{2m+1}.$$

66. 1° Let us write our sum in the following way

$$s = 1\cdot C_n^0 + 2C_n^1 + 3C_n^2 + \dots + (n+1)C_n^n = \sum_{k=0}^{k=n}(k+1)C_n^k,$$

and introduce a new summation variable. Put $k = n - k'$. Then the sum is rewritten as

$$s = \sum_{k'=n}^{k'=0}(n-k'+1)C_n^{n-k'} = \sum_{k=0}^{k=n}(n-k+1)C_n^k =$$

$$= \sum_{k=0}^{k=n}[n+2-(k+1)]C_n^k =$$

$$= (n+2)\sum_{k=0}^{k=n}C_n^k - \sum_{k=0}^{k=n}(k+1)C_n^k = (n+2)2^n - s.$$

Consequently,

$$2s=(n+2)\,2^n,\qquad s=(n+2)\,2^{n-1}.$$

This sum can be computed in a somewhat different way. Rewrite it as follows

$$s=(C_n^0+C_n^1+\ldots+C_n^n)+(C_n^1+2C_n^2+\ldots+nC_n^n)=2^n+n+$$
$$+2\frac{n(n-1)}{1\cdot 2}+3\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\ldots+n(n-1)+n\cdot 1=$$
$$=2^n+n\left\{1+(n-1)+\frac{(n-1)(n-2)}{1\cdot 2}+\ldots+(n-1)+1\right\}=$$
$$=2^n+n\{C_{n-1}^0+C_{n-1}^1+\ldots+C_{n-1}^{n-1}\}=2^n+n2^{n-1}=2^{n-1}(n+2).$$

2° We have

$$C_n^1-2C_n^2+3C_n^3+\ldots+(-1)^{n-1}nC_n^n=n-2\frac{n(n-1)}{1\cdot 2}+$$
$$+3\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\ldots+(-1)^{n-1}n=$$
$$=n\left\{1-\frac{n-1}{1}+\frac{(n-1)(n-2)}{1\cdot 2}+\ldots+(-1)^{n-2}\frac{n-1}{1}+\right.$$
$$\left.+(-1)^{n-1}\right\}=n(1-1)^{n-1}=0.$$

67. Rewrite the sum in the following manner

$$\frac{1}{2}C_n^1-\frac{1}{3}C_n^2+\frac{1}{4}C_n^3-\ldots+\frac{(-1)^{n-1}}{n+1}C_n^n=$$
$$=\frac{n}{2}-\frac{n(n-1)}{1\cdot 2\cdot 3}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}+\ldots+\frac{(-1)^{n-1}}{n+1}=$$
$$=\frac{1}{n+1}\left\{\frac{(n+1)n}{1\cdot 2}-\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}+\ldots+(-1)^{n-1}\right\}=$$
$$=\frac{1}{n+1}\left\{\left[1-\frac{n+1}{1}+\frac{(n+1)n}{1\cdot 2}-\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}+\ldots+\right.\right.$$
$$\left.\left.+(-1)^{n+1}\right]-1+\frac{n+1}{1}\right\}=\frac{1}{n+1}\{(1-1)^{n+1}+n\}=\frac{n}{n+1}.$$

68. 1° Consider the following polynomial

$$(1+x)^{n+1}=1+C_{n+1}^1x+C_{n+1}^2x^2+\ldots+C_{n+1}^{n+1}x^{n+1}.$$

Hence

$$\frac{(1+x)^{n+1}-1}{n+1}=C_n^0x+\frac{C_n^1}{2}x^2+\frac{C_n^2}{3}x^3+\ldots+\frac{C_n^n}{n+1}x^{n+1}.$$

Putting $x=1$, we get the required identity.

2° Obtained from the preceding identity at $x=2$.

69. Put

$$C_n^1-\frac{1}{2}C_n^2+\frac{1}{3}C_n^3+\ldots+\frac{(-1)^{n-1}}{n}C_n^n=u_n.$$

Then we have

$$u_n-u_{n-1}=\left\{n-\frac{1}{2}\frac{n(n-1)}{1\cdot 2}+\frac{1}{3}\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\ldots\right\}-$$

$$-\left\{n-1-\frac{1}{2}\frac{(n-1)(n-2)}{1\cdot 2}+\frac{1}{3}\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}-\ldots\right\}=$$

$$=\{n-(n-1)\}-\frac{1}{2}\left\{\frac{n(n-1)}{1\cdot 2}-\frac{(n-1)(n-2)}{1\cdot 2}\right\}+$$

$$+\frac{1}{3}\left\{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}-\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}\right\}+\ldots=$$

$$=1-\frac{n-1}{1\cdot 2}+\frac{(n-1)(n-2)}{1\cdot 2\cdot 3}+\ldots=$$

$$=\frac{1}{n}\left\{n-\frac{n(n-1)}{1\cdot 2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}-\ldots\right\}=$$

$$=\frac{1}{n}\{1-(1-1)^n\}=\frac{1}{n}.$$

And so,

$$u_n-u_{n-1}=\frac{1}{n}.$$

Therefore we may write a number of equalities

$$u_2-u_1=\frac{1}{2},$$

$$u_3-u_2=\frac{1}{3},$$

$$\cdots\cdots\cdots$$

$$u_n-u_{n-1}=\frac{1}{n}.$$

Adding them term by term, we find

$$u_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

70. 1° We may proceed as follows. The expression on the left is the coefficient of x^n in the following polynomial

$$s=(1+x)^n+(1+x)^{n+1}+(1+x)^{n+2}+\ldots+(1+x)^{n+k},$$

Transforming this polynomial, we have

$$s=(1+x)^n\{1+(1+x)+(1+x)^2+\ldots+(1+x)^k\}=$$
$$=(1+x)^n\frac{(1+x)^{k+1}-1}{x}=\frac{1}{x}\{(1+x)^{n+k+1}-(1+x)^n\}.$$

The coefficient of x^{n+1} in the braced polynomial is equal to C_{n+k+1}^{n+1}. Thus, our proposition is proved.

2° The expression on the left is the coefficient of x^n in the following polynomial

$$x^n(1+x)^n-x^{n-1}(1+x)^n+x^{n-2}(1+x)^n+\ldots+$$
$$+(-1)^h x^{n-h}(1+x)^n=(1+x)^n\{x^n-x^{n-1}+\ldots+$$
$$+(-1)^h x^{n-h}\}=(1+x)^{n-1}\{x^{n+1}+(-1)^h x^{n-h}\}.$$

It is obvious that the coefficient of x^n in the last expression is equal to

$$(-1)^h C_{n-1}^h.$$

71. 1° Consider the following polynomials

$$(1+x)^n=\sum_{s=0}^{n} C_n^s x^s,\quad (1+x)^m=\sum_{t=0}^{m} C_m^t x^t.$$

We have

$$(1+x)^n(1+x)^m=\sum_{s=0}^{n} C_n^s x^s \sum_{t=0}^{m} C_m^t x^t=$$
$$=(1+x)^{m+n}=\sum_{p=0}^{m+n} C_{m+n}^p x^p,$$

wherefrom follows the required equality.

2° Follows from 1°.

72. 1° Consider the product

$$(1+x)^n(1+x)^n=(1+x)^{2n}.$$

We have

$$\sum_{s=0}^{n} C_n^s x^s \sum_{t=0}^{n} C_n^t x^t=\sum_{l=0}^{2n} C_{2n}^l x^l.$$

Hence

$$C_{2n}^l=\sum_{s+t=l} C_n^s C_n^t.$$

Consequently

$$C_{2n}^n = \sum_{s+t=n} C_n^s \cdot C_n^t = \sum_{s=0}^{n} C_n^s C_n^{n-s} = \sum_{s=0}^{n} (C_n^s)^2.$$

2° In this case we consider the following product

$$(1+x)^m (1-x)^m = (1-x^2)^m. \qquad (*)$$

Consequently

$$\sum_{s=0}^{m} (-1)^s C_m^s x^s \sum_{t=0}^{m} C_m^t x^t = \sum_{l=0}^{m} (-1)^l C_m^l x^{2l},$$

therefore

$$\sum_{s+t=2l} (-1)^s C_m^s C_m^t = (-1)^l C_m^l.$$

Let us assume first that m is even and put $m=2n$. Let $l=n$. Then

$$\sum_{s+t=2n} (-1)^s C_{2n}^s C_{2n}^t = (-1)^n C_{2n}^n.$$

Hence

$$\sum_{s=0}^{2n} (-1)^s (C_{2n}^s)^2 = (-1)^n C_{2n}^n.$$

3° If m is odd, then we put $m = 2n+1$. The coefficient of x^{2n+1} in the left member of the equality (*) is equal to

$$\sum_{s+t=2n+1} (-1)^s C_{2n+1}^s C_{2n+1}^t = \sum_{s=0}^{2n+1} (-1)^s (C_{2n+1}^s)^2.$$

But the right member of the equality (*) shows that this coefficient must equal zero (since it is evident from the expansion that odd powers of x are absent). Therefore

$$\sum_{s=0}^{2n+1} (-1)^s (C_{2n+1}^s)^2 = 0$$

and equality 3° is proved.

4° We have two equalities

$$C_n^1 x + 2C_n^2 x^2 + \ldots + nC_n^n x^n = nx(1+x)^{n-1},$$

$$C_n^0 + C_n^1 x + \ldots + C_n^n x^n = (1+x)^n.$$

Multiplying them termwise, we find

$$\sum_{s=0}^{n} sC_n^s x^s \sum_{k=0}^{n} C_n^k x^k = nx(1+x)^{2n-1}.$$

Equating the coefficients of x^n in both members of these equalities, we get the required identity.

73. Since the product $(x-a)(x-b)$ is a second-degree trinomial, when divided by it, the polynomial $f(x)$ will necessarily leave a remainder which is a first-degree polynomial in x, $\alpha x+\beta$. Thus, there exists the following identity

$$f(x) = (x-a)(x-b)Q(x) + \alpha x + \beta.$$

It only remains to determine α and β. Putting in this identity first $x = a$ and then $x = b$, we get

$$f(a) = \alpha a + \beta,$$
$$f(b) = \alpha b + \beta.$$

But we know that the remainder from dividing $f(x)$ by $x-a$ is equal to $f(a)$, therefore,

$$f(a) = A,$$
$$f(b) = B.$$

Thus, for determining α and β we get the following system of two equations in two unknowns

$$\alpha a + \beta = A$$
$$\alpha b + \beta = B.$$

Hence

$$\alpha = \frac{1}{a-b}(A-B),$$
$$\beta = \frac{aB-bA}{a-b}.$$

74. Reasoning as in the preceding problem, we conclude that the remainder will have the following form

$$\alpha x^2 + \beta x + \gamma.$$

For determining α, β and γ we have the following system

$$\alpha a^2 + \beta a + \gamma = A$$
$$\alpha b^2 + \beta b + \gamma = B$$
$$\alpha c^2 + \beta c + \gamma = C.$$

On determining α, β and γ, we may represent the required remainder $\alpha x^2 + \beta x + \gamma$ in the following symmetric form

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} A + \frac{(x-a)(x-c)}{(b-a)(b-c)} B + \frac{(x-a)(x-b)}{(c-a)(c-b)} C.$$

75. The remainder will be

$$\frac{(x-x_2)(x-x_3)\dots(x-x_m)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_m)} y_1 + \frac{(x-x_1)(x-x_3)\dots(x-x_m)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_m)} y_2 + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{m-1})}{(x_m-x_1)(x_m-x_2)\dots(x_m-x_{m-1})} y_m.$$

76. The required polynomial (see the preceding problem) takes the form

$$\frac{(x-a_2)(x-a_3)\dots(x-a_m)}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_m)} A_1 + \frac{(x-a_1)(x-a_3)\dots(x-a_m)}{(a_2-a_1)(a_2-a_3)\dots(a_2-a_m)} A_2 + \dots + \frac{(x-a_1)(x-a_2)\dots(x-a_{m-1})}{(a_m-a_1)(a_m-a_2)\dots(a_m-a_{m-1})} A_m.$$

77. Our equality states the identity of two polynomials. For this purpose it is sufficient to establish that the polynomial

$$f(x_1)\frac{(x-x_2)(x-x_3)\dots(x-x_m)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_m)} + f(x_2)\frac{(x-x_1)(x-x_3)\dots(x-x_m)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_m)} + \dots + f(x_m)\frac{(x-x_1)(x-x_2)\dots(x-x_{m-1})}{(x_m-x_1)\dots(x_m-x_{m-1})} - f(x)$$

is identically equal to zero. Since the degree of this polynomial is equal to $m-1$, it suffices to establish that it vanishes at m different values of x. Indeed, it is easy to check that this polynomial is really equal to zero at

$$x = x_1,\ x_2,\ x_3,\ \ldots,\ x_m.$$

78. Obtained from the previous problem by equating the coefficients of x^{m-1}.

79. If we put in the preceding problem $f(x) = 1,\ x,\ x^2,\ \ldots,\ x^{m-2}$, then it will be proved that $s_n = 0$ if $0 \leqslant \leqslant n < m - 1$. To prove the identity

$$s_{m-1} = 1$$

it is sufficient to put $f(x) = x^{m-1}$ in the identity of Problem 77 and to equate the coefficients of x^{m-1} in both members of the identity being obtained. To compute s_n for $n > > m - 1$ it is possible to proceed in the following way. Suppose $x_1, x_2, \ldots, x_m$ satisfy an equation of degree m

$$\alpha^m + p_1\alpha^{m-1} + p_2\alpha^{m-2} + \ldots + p_{m-1}\alpha + p_m = 0,$$

where

$$-p_1 = x_1 + x_2 + \ldots + x_m,$$
$$p_2 = x_1x_2 + x_2x_3 + \ldots + x_{m-1}x_m,$$
$$-p_3 = x_1x_2x_3 + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(-1)^k p_k = x_1x_2 \ldots x_k + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Multiplying both members of our equation by α^k, we get

$$\alpha^{m+k} + p_1\alpha^{m+k-1} + p_2\alpha^{m+k-2} + \ldots + p_{m-1}\alpha^{k+1} + p_m\alpha^k = 0.$$

Putting in this equality successively $\alpha = x_1, x_2, \ldots, x_m$ and adding, we find

$$s_{m+k} + p_1s_{m+k-1} + p_2s_{m+k-2} + \ldots + p_{m-1}s_{k+1} + p_ms_k = 0.$$

At $k = 0$ we have

$$s_m + p_1s_{m-1} = 0.$$

Consequently

$$s_m = -p_1 = x_1 + x_2 + \ldots + x_m.$$

At $k = 1$ we obtain

$$s_{m+1} + p_1s_m + p_2s_{m-1} = 0.$$

Further

$$s_{m+1} = (x_1 + x_2 + x_3 + \ldots + x_m)^2 - \\ - (x_1x_2 + \ldots + x_{m-1}x_m) = \\ = x_1^2 + x_2^2 + \ldots + x_m^2 + x_1x_2 + x_1x_3 + \ldots,$$

i.e. s_{m+1} is equal to a sum of products of the factors

$$x_1, x_2, \ldots, x_m$$

taken pairwise.

Here the factors may be both equal and unequal. Similar results can be obtained for s_{m+2}, s_{m+3} and so on. The same results can be obtained using a more elegant method (Gauss, *Theoria interpolationis methodo nova tractata*). Put

$$\frac{1}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)} = \alpha_1$$

$$\frac{1}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)} = \alpha_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})} = \alpha_m.$$

Then we have

$$s_n = x_1^n\alpha_1 + x_2^n\alpha_2 + \ldots + x_m^n\alpha_m.$$

Let us form the following expression

$$P = \frac{\alpha_1}{1-x_1z} + \frac{\alpha_2}{1-x_2z} + \ldots + \frac{\alpha_m}{1-x_mz}. \qquad (*)$$

Using the formula for an infinitely decreasing geometric progression and assuming that z is chosen so that $|x_1z| < 1$, $|x_2z| < 1, \ldots, |x_mz| < 1$, expand the sum in an infinite series in the following way

$$P = \alpha_1(1 + x_1z + x_1^2z^2 + x_1^3z^3 + \ldots) + \alpha_2(1 + x_2z + x_2^2z^2 + \\ + x_2^3z^3 + \ldots) + \ldots + \alpha_m(1 + x_mz + x_m^2z^2 + x_m^3z^3 + \ldots).$$

Or

$$P = (\alpha_1 + \alpha_2 + \ldots + \alpha_m) + (x_1\alpha_1 + x_2\alpha_2 + \ldots + x_m\alpha_m)z + \\ + (x_1^2\alpha_1 + x_2^2\alpha_2 + \ldots + x_m^2\alpha_m)z^2 + \ldots,$$

i.e.

$$P = s_0 + s_1z + s_2z^2 + s_3z^3 + \ldots.$$

Put for brevity

$$(1-x_1z)(1-x_2z)\ldots(1-x_mz)=Q.$$

Expanding Q in powers of z, we can write

$$Q=1-\sigma_1z+\sigma_2z^2+\ldots\pm\sigma_mz^m,$$

where

$$\sigma_1=x_1+x_2+\ldots+x_m,$$
$$\sigma_2=x_1x_2+x_1x_3+\ldots+x_{m-1}x_m,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Multiplying both members of (*) by $(1-x_1z)(1-x_2z)\ldots\times\times(1-x_mz)$, we have

$$PQ=\alpha_1(1-x_2z)(1-x_3z)\ldots(1-x_mz)+$$
$$+\alpha_2(1-x_1z)(1-x_3z)\ldots(1-x_mz)+$$
$$+\alpha_3(1-x_1z)(1-x_2z)(1-x_3z)\ldots(1-x_mz)+\ldots+$$
$$+\alpha_m(1-x_1z)(1-x_2z)\ldots(1-x_{m-1}z).$$

Thus, the product PQ is an $(m-1)$th-degree polynomial in z. Let us show that it is simply equal to z^{m-1}, i.e. the following identity takes place

$$PQ=z^{m-1}.$$

Indeed, the expression $PQ-z^{m-1}$ becomes zero at $z=\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m}$. At $z=\frac{1}{x_1}$ we have

$$\alpha_1\left(1-\frac{x_2}{x_1}\right)\left(1-\frac{x_3}{x_1}\right)\ldots\left(1-\frac{x_m}{x_1}\right)-\frac{1}{x_1^{m-1}}=$$
$$=\frac{1}{x_1^{m-1}}-\frac{1}{x_1^{m-1}}=0.$$

Let us show in the same way that $PQ-z^{m-1}$ vanishes at $z=\frac{1}{x_2}, \ldots, \frac{1}{x_m}$. But if a polynomial of degree $m-1$ vanishes at m different values of the variable, then it is identically equal to zero. Thus, $PQ-z^{m-1}\equiv 0$. Consequently

$$\frac{z^{m-1}}{Q}=P.$$

Or

$$z^{m-1}\frac{1}{1-\sigma_1 z+\sigma_2 z^2-\sigma_3 z^3+\ldots\pm\sigma_m z^m}=$$

$$=s_0+s_1 z+\ldots+s_{m-2}z^{m-2}+s_{m-1}z^{m-1}+\ldots.$$

If we expand the left member in an infinite series in powers of z, then this series will begin only with a term containing z^{m-1}. Therefore the coefficients of z^0, z^1, ..., z^{m-2} must also be equal to zero on the right, i.e. we have

$$s_0=s_1=s_2=\ldots=s_{m-2}=0.$$

Besides, the coefficient at z^{m-1} in the left member is equal to 1. Therefore

$$s_{m-1}=1.$$

Now our equality takes the following form

$$\frac{z^{m-1}}{1-\sigma_1 z+\sigma_2 z^2-\sigma_3 z^3+\ldots\pm\sigma_m z^m}=z^{m-1}+s_m z^m+s_{m+1}z^{m+1}+\ldots.$$

Reducing both members by z^{m-1}, we find

$$\frac{1}{1-\sigma_1 z+\sigma_2 z^2-\sigma_3 z^3+\ldots\pm\sigma_m z^m}=1+s_m z+s_{m+1}z^2+\ldots$$

or

$$1=(1-\sigma_1 z+\sigma_2 z^2-\sigma_3 z^3+\ldots\pm\sigma_m z^m)(1+s_m z+$$
$$+s_{m+1}z^2+\ldots).$$

Arranging the right member in powers of z and equating the coefficients of these powers to zero (since the left member contains only 1), we find

$$s_m-\sigma_1=0,$$
$$\sigma_2-\sigma_1 s_m+s_{m+1}=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus, we get a possibility to compute s_m, s_{m+1}, s_{m+2}, ... However, to determine the general structure of s_{m+1}, let us consider

$$\frac{1}{Q}=\frac{1}{1-x_1 z}\cdot\frac{1}{1-x_2 z}\cdots\frac{1}{1-x_m z}=\sum_{s=0}^{\infty}x_1^s z^s\sum_{s'=0}^{\infty}x_2^{s'}z^{s'}\ldots=$$

$$=\sum x_1^s x_2^{s'} x_3^{s''}\ldots z^{s+s'+s''}\ldots.$$

But, on the other hand,

$$\frac{1}{Q}=1+s_m z+s_{m+1}z^2+\ldots+s_{m+k}z^{k+1}+\ldots,$$

therefore we get

$$s_{m+k}=\sum_{s+s'+s''+\ldots=k+1} x_1^s x_2^{s'} x_3^{s''}\ldots .$$

Thus, we get the following final result: s_{m+k} *is equal to a sum of products of* $k+1$ *equal or unequal quantities taken from the totality* $x_1, x_2, \ldots, x_m$. In particular

$$s_{m+1}=x_1^2+x_2^2+\ldots+x_n^2+x_1x_2+x_1x_3+\ldots+ \\ +x_1x_m+x_2x_3+\ldots+x_{m-1}x_m,$$

$$s_{m+2}=x_1^3+x_2^3+\ldots+x_m^3+x_1^2x_2+\ldots+x_{m-1}^2x_m+x_1x_2x_3+\ldots .$$

80. Let us introduce the following notation

$$s_n(x_1, x_2, \ldots, x_m)=\frac{x_1^n}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)}+ \\ +\frac{x_2^n}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)}+\ldots+ \\ +\frac{x_m^n}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})}.$$

We have

$$\frac{x_1^{-n}}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)}= \\ =\frac{x_1^{-n-m+2}}{x_1x_2\ldots x_m\left(\frac{1}{x_2}-\frac{1}{x_1}\right)\left(\frac{1}{x_3}-\frac{1}{x_1}\right)\ldots\left(\frac{1}{x_m}-\frac{1}{x_1}\right)}= \\ =(-1)^{m-1}\frac{\left(\frac{1}{x_1}\right)^{n+m-2}}{\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\left(\frac{1}{x_1}-\frac{1}{x_3}\right)\ldots\left(\frac{1}{x_1}-\frac{1}{x_m}\right)}\cdot\frac{1}{x_1x_2\ldots x_m},$$

therefore it is obvious that

$$s_{-n}(x_1, x_2, \ldots, x_m)=\frac{(-1)^{m-1}}{x_1x_2\ldots x_m}s_{n+m-2}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m}\right).$$

81. The validity of the assertion follows from the identity of Problem 77. The same identity yields

$$A_1=\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_m)},$$

$$A_2=\frac{f(x_2)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_m)},$$

. . . .

$$A_m=\frac{f(x_m)}{(x_m-x_1)(x_m-x_2)\ldots(x_m-x_{m-1})}.$$

82. Set up the expression

$$\frac{x_1}{\lambda-b_1}+\frac{x_2}{\lambda-b_2}+\ldots+\frac{x_n}{\lambda-b_n}=$$
$$=1-\frac{(\lambda-a_1)(\lambda-a_2)\ldots(\lambda-a_n)}{(\lambda-b_1)(\lambda-b_2)\ldots(\lambda-b_n)}. \quad (*)$$

If all the terms are transposed to the left and reduced to a common denominator and then the latter is removed, then the left member becomes a polynomial in λ of degree $n-1$.

By virtue of existence of the given system of equations this polynomial vanishes at n different values of λ, namely at $\lambda=a_1, a_2, \ldots, a_n$. Therefore it is identically equal to zero, and, consequently, the original equality (*) is also an identity. But then the equality (*) represents an expansion into partial fractions of the following fraction

$$\frac{(\lambda-b_1)(\lambda-b_2)\ldots(\lambda-b_n)-(\lambda-a_1)(\lambda-a_2)\ldots(\lambda-a_n)}{(\lambda-b_1)(\lambda-b_2)\ldots(\lambda-b_n)}.$$

Therefore, the unknowns $x_1, x_2, \ldots, x_n$ are found by the formulas of the preceding problem, and we get

$$x_1=-\frac{(b_1-a_1)(b_1-a_2)\ldots(b_1-a_n)}{(b_1-b_2)(b_1-b_3)\ldots(b_1-b_n)},$$
$$x_2=-\frac{(b_2-a_1)(b_2-a_2)\ldots(b_2-a_n)}{(b_2-b_1)(b_2-b_3)\ldots(b_2-b_n)}, \ \ldots .$$

83. Readily obtained by applying the result of Problem 81.

84. Consider the following fraction

$$\frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(x-b_1)(x-b_2)\ldots(x-b_n)}.$$

It is obvious that the difference

$$\frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(x-b_1)(x-b_2)\ldots(x-b_n)}-1,$$

on reducing to a common denominator, will be a fraction in which the power of the numerator is less than that of the denominator. This fraction can be expanded into partial fractions. Therefore, the following identity takes place

$$\frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(x-b_1)(x-b_2)\ldots(x-b_n)}=1+\frac{A_1}{x-b_1}+\frac{A_2}{x-b_2}+\ldots+\frac{A_n}{x-b_n}.$$

Multiplying both members of this identity by $x-b_1$, we find

$$\frac{(x-a_1)(x-a_2)\dots(x-a_n)}{(x-b_2)(x-b_3)\dots(x-b_n)}=x-b_1+A_1+$$
$$+\frac{A_2}{x-b_2}(x-b_1)+\dots+\frac{A_n}{x-b_n}(x-b_1).$$

In this identity we may put

$$x=b_1.$$

We then have

$$A_1=\frac{(b_1-a_1)(b_1-a_2)\dots(b_1-a_n)}{(b_1-b_2)(b_1-b_3)\dots(b_1-b_n)}.$$

Similar expressions are obtained for A_2, A_3, ..., A_n.

Thus, we have the following identity

$$\frac{(x-a_1)(x-a_2)\dots(x-a_n)}{(x-b_1)(x-b_2)\dots(x-b_n)}=1+\frac{(b_1-a_1)(b_1-a_2)\dots(b_1-a_n)}{(b_1-b_2)(b_1-b_3)\dots(b_1-b_n)}\times$$
$$\times\frac{1}{x-b_1}+\frac{(b_2-a_1)(b_2-a_2)\dots(b_2-a_n)}{(b_2-b_1)(b_2-b_3)\dots(b_2-b_n)}\cdot\frac{1}{x-b_2}+\dots+$$
$$+\frac{(b_n-a_1)(b_n-a_2)\dots(b_n-a_n)}{(b_n-b_1)(b_n-b_2)\dots(b_n-b_{n-1})}\cdot\frac{1}{x-b_n}.$$

At $x=0$ we get the required identity.

85. As in the preceding problem, it is easy to see that

$$\frac{(x+\beta)(x+2\beta)\dots(x+n\beta)}{(x-\beta)(x-2\beta)\dots(x-n\beta)}=1+\sum_{r=1}^{n}\frac{A_r}{x-r\beta}.$$

where

$$A_r=\frac{(r\beta+\beta)(r\beta+2\beta)\dots(r\beta+n\beta)}{(r\beta-\beta)(r\beta-2\beta)\dots[r\beta-(r-1)\beta][r\beta-(r+1)\beta]\dots(r\beta-n\beta)}.$$

It only remains to simplify this coefficient.

86. We have

$$c_{k+1}-c_k=\Delta c_k,\text{ i.e. } c_{k+1}=c_k+\Delta c_k.$$

and formula 1° holds at $n = 1$. Assuming that it is true at n, let us prove its validity at $n + 1$. Indeed

$$c_{k+n+1} = c_{k+n} + \Delta c_{k+n} =$$

$$= \left(c_k + \frac{n}{1}\Delta c_k + \frac{n(n-1)}{1\cdot 2}\Delta^2 c_k + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\Delta^3 c_k + \ldots + \right.$$

$$\left. + \Delta^n c_k\right) + \Delta\left(c_k + \frac{n}{1}\Delta c_k + \frac{n(n-1)}{1\cdot 2}\Delta^2 c_k + \ldots + \Delta^n c_k\right) =$$

$$= c_k + \left(\frac{n}{1} + 1\right)\Delta c_k + \left(\frac{n(n-1)}{1\cdot 2} + \frac{n}{1}\right)\Delta^2 c_k + \ldots + \Delta^{n+1} c_k =$$

$$= c_k + \frac{n+1}{1}\Delta c_k + \frac{(n+1)n}{1\cdot 2}\Delta^2 c_k + \ldots + \Delta^{n+1} c_k,$$

and the proposition is proved.

Formula 2° is proved likewise. It is obvious that at $n = 1$ it holds true. Let us assume that it is valid at n. Then we have

$$\Delta^{n+1} c_k = \Delta c_{k+n} - \frac{n}{1}\Delta c_{k+n-1} + \frac{n(n-1)}{1\cdot 2}\Delta c_{k+n-2} - \ldots +$$

$$+ (-1)^n \Delta c_k = (c_{k+n+1} - c_{k+n}) - \frac{n}{1}(c_{k+n} - c_{k+n-1}) +$$

$$+ \frac{n(n-1)}{1\cdot 2}(c_{k+n-1} - c_{k+n-2}) + \ldots + (-1)^n (c_{k+1} - c_k) =$$

$$= c_{k+n+1} - \frac{n+1}{1} c_{k+n} + \frac{(n+1)n}{1\cdot 2} c_{k+n-1} - \ldots + (-1)^{n+1} c_k.$$

87. It is not difficult to check the validity of this formula. We see that the right member is an nth-degree polynomial in x. Let us designate it by $\varphi(x)$, i.e. let us put

$$f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1\cdot 2}\Delta^2 f(0) + \ldots +$$

$$+ \frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n f(0) = \varphi(x).$$

Let in this equality $x = 0$. We get $\varphi(0) = f(0)$, at $x = 1$ we find

$$\varphi(1) = f(0) + \Delta f(0) = f(1).$$

Using formula 1° of the preceding problem we may state that in general

$$\varphi(k) = f(k) \text{ at } k = 0, 1, 2, \ldots, n.$$

Thus, two polynomials [$\varphi(x)$ and $f(x)$] of degree n are equal to each other at $n + 1$ different values of the independent variable x, consequently, they are equal identically, and we have

$$\varphi(x) = f(x)$$

for any x.

And so, we have checked the validity of the formulas. It is not difficult to deduce this formula.

Let $f(x)$ be an nth-degree polynomial. First of all we assert that it is always possible to choose the coefficients $A_0, A_1, A_2, \ldots, A_n$ such that the following identity takes place

$$f(x) = A_0 - A_1x + A_2x(x-1) + A_3x(x-1)(x-2) + \\ + \ldots + A_nx(x-1)(x-2)\ldots(x-n+1).$$

Indeed, let us divide the polynomial $f(x)$ by $(x-1) \times \times (x-2) \ldots (x-n)$. Since the last polynomial is also of degree n, the quotient will be a constant, and the remainder a polynomial of degree not exceeding $n - 1$. Dividing this polynomial by $x(x-1)\ldots(x-n+1)$, we find the constant A_{n-1} and so on.

Let us now compute the constants $A_0, A_1, A_2, \ldots, A_{n-1}, A_n$.

Put for brevity

$$x(x-1)(x-2)\ldots(x-k+1) = \varphi_k(x) \qquad (k = 1, 2, 3, \ldots).$$

Then we have

$$\Delta\varphi_k(x) = \varphi_k(x-1) - \varphi_k(x) = \\ = (x+1)x(x-1)\ldots(x-k+2) - \\ - x(x-1)\ldots(x-k+1) = \\ = k \cdot x(x-1)\ldots(x-k+2) = k\varphi_{k-1}(x).$$

To determine $A_0, A_1, A_2, \ldots, A_n$ proceed in the following way. Put in our identity $x = 0$. Since $\varphi_k(0) = 0$, we find

$$A_0 = f(0).$$

Let us now take the difference between the members of the identity. We obtain

$$\Delta f(x) = A_1 \Delta\varphi_1(x) + A_2 \Delta\varphi_2(x) + \dots + A_n \Delta\varphi_n(x) = A_1 + 2A_2\varphi_1(x) + \dots + nA_n\varphi_{n-1}(x).$$

Putting here $x = 0$, we have

$$A_1 = \Delta f(0).$$

Further

$$\Delta^2 f(x) = 2A_2\Delta\varphi_1(x) + \dots + nA_n \Delta\varphi_{n-1}(x) = 2A_2 + \dots + n(n-1) A_n\varphi_{n-2}(x).$$

Hence

$$A_2 = \frac{\Delta^2 f(0)}{1\cdot 2}.$$

Continuing this operation, we find all the coefficients

$$A_0, A_1, \dots, A_n.$$

88. Replacing x by $x+1$, we have

$$(x+1)^n = A_0 + A_1 x + \frac{A_2}{2!}x(x-1) + \frac{A_3}{3!}x(x-1)(x-2) + \dots + \frac{A_n}{n!}x(x-1)\dots(x-n+1).$$

Putting $f(x) = (x+1)^n$ and using the result of the preceding problem, we find

$$A_s = \Delta^s f(0).$$

From formula 2° of Problem 86 we get the required expression for A_s.

89. Putting $k = 0$ in formula 2° of Problem 86, we get

$$\Delta^n c_0 = c_n - \frac{n}{1}c_{n-1} + \frac{n(n-1)}{1\cdot 2}c_{n-2} - \dots + (-1)^n c_0.$$

Put

$$c_0 = \frac{1}{(x+n)^2}$$

and take

$$c_0 = \frac{1}{(x+n)^2}, \quad c_1 = \frac{1}{(x+n-1)^2}, \quad \dots, \quad c_n = \frac{1}{x^2};$$

to prove our identity it only remains to prove that

$$\Delta^n \frac{1}{(x+n)^2} = \frac{n!}{x(x+1)\dots(x+n)} \left\{ \frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+n} \right\}.$$

Use the method of induction. At $n=1$ the formula is true. Assuming, as usual, its validity for n, let us prove that it is also valid for $n+1$. We have

$$\Delta^{n+1} \frac{1}{(x+n+1)^2} = \Delta\left(\Delta^n \frac{1}{(x+n+1)^2}\right) =$$

$$= \Delta\left\{ \frac{n!}{(x+1)(x+2)\dots(x+n+1)} \left(\frac{1}{x+1} + \frac{1}{x+2} + \dots + \frac{1}{x+n+1} \right) \right\} = \frac{n!}{x(x+1)\dots(x+n)} \left\{ \frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+n} \right\} -$$

$$- \frac{n!}{(x+1)(x+2)\dots(x+n+1)} \left\{ \frac{1}{x+1} + \frac{1}{x+2} + \dots + \frac{1}{x+n+1} \right\} =$$

$$= \frac{n!}{x(x+1)\dots(x+n+1)} \left\{ (x+n+1)\left(\frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+n} \right) - x\left(\frac{1}{x+1} + \frac{1}{x+2} + \dots + \frac{1}{x+n+1} \right) \right\} =$$

$$= \frac{(n+1)!}{x(x+1)\dots(x+n+1)} \left\{ \frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+n+1} \right\}.$$

At $x=1$ our identity yields

$$\frac{1}{n+1} \left\{ \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n+1} \right\} = \frac{1}{1^2} - \frac{C_n^1}{2^2} + \dots + (-1)^n \frac{1}{(n+1)^2}.$$

90. The expression $\varphi_n(x+y)$ is an nth-degree polynomial in x. Therefore we may represent it as (see Problem 87)

$$\varphi_n(x+y) = A_0 + A_1\varphi_1(x) + A_2\varphi_2(x) + \dots + A_n\varphi_n(x),$$

where $A_s = \frac{\Delta^s \varphi_n(y)}{s!}$ (since $\varphi_n(x+y)$ turns into $\varphi_n(y)$ at $x=0$). However, it is known (Problem 87) that $\Delta\varphi_n(y) = n\varphi_{n-1}(y)$, consequently

$$\Delta^2\varphi_n(y) = n(n-1)\varphi_{n-2}(y),$$

$$\dots\dots\dots\dots\dots\dots$$

$$\Delta^s\varphi_n(y) = n(n-1)\dots(n-s+1)\varphi_{n-s}(y).$$

Thus

$$A_s = \frac{n(n-1)(n-2)\dots(n-s+1)\varphi_{n-s}(y)}{s!} = C_n^s\varphi_{n-s}(y),$$

and our formula is valid.

However, the validity of this formula can be proved using other reasons. Let x and y be positive integers greater than n. Then the following equalities take place

$$(1+z)^x = 1 + xz + \frac{x(x-1)}{1\cdot 2}z^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}z^3 + \dots,$$

$$(1+z)^y = 1 + yz + \frac{y(y-1)}{1\cdot 2}z^2 + \frac{y(y-1)(y-2)}{1\cdot 2\cdot 3}z^3 + \dots,$$

$$(1+z)^{x+y} = 1 + (x+y)z + \frac{(x+y)(x+y-1)}{1\cdot 2}z^2 +$$

$$+ \frac{(x+y)(x+y-1)(x+y-2)}{1\cdot 2\cdot 3}z^3 + \dots.$$

On the other hand,

$$(1+z)^x\cdot(1+z)^y = (1+z)^{x+y},$$

i. e.

$$\sum \frac{\varphi_k(x)}{k!}z^k\cdot\sum\frac{\varphi_s(y)}{s!}z^s = \sum\frac{\varphi_n(x+y)}{n!}z^n.$$

Equating the coefficients of z^n in both members of this equality, we get

$$\varphi_n(x+y) = \varphi_n(x) + C_n^1\varphi_{n-1}(x)\varphi_1(y) + \dots +$$

$$+ C_n^{n-1}\varphi_1(x)\varphi_{n-1}(y) + \varphi_n(y).$$

Let y_0 be a whole positive number exceeding n. Then

$$\varphi_n(x+y_0) \text{ and } \varphi_n(x) + C_n^1\varphi_{n-1}(x)\varphi_1(y_0) + \dots + \varphi_n(y_0)$$

are two nth-degree polynomials in x, and they are equal to each other at all whole values of x exceeding n. Consequently, they equal identically at all values of x. But y_0 may attain all whole values exceeding n. Consequently, as in the previous case, we conclude that y_0 can attain any values and the equality

$$\varphi_n(x+y) = \varphi_n(x) + C_n^1\varphi_{n-1}(x)\varphi_1(y) + \dots +$$

$$+ C_n^{n-1}\varphi_1(x)\varphi_{n-1}(y) + \varphi_n(y)$$

is valid for any values of x and y.

91. First of all, both identities 1° and 2° can be readily proved using the method of mathematical induction. Indeed, at $n = 1$ identity 1° takes place. Suppose it takes place for all values of the exponent, not exceeding n, so that we have

$$x^n + y^n = p^n - \frac{n}{1} p^{n-2}q + \frac{n(n-3)}{1\cdot 2} p^{n-4}q^2 - \\ - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3} p^{n-6}q^3 + \ldots .$$

Multiplying both members of this equality by $x + y = p$, we get

$$x^{n+1} + y^{n+1} + xy(x^{n-1} + y^{n-1}) = \\ = p^{n+1} - \frac{n}{1} p^{n-1}q + \frac{n(n-3)}{1\cdot 2} p^{n-3}q^2 - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3} \times \\ \times p^{n-5}q^3 + \ldots .$$

Hence

$$x^{n+1} + y^{n+1} = \\ = p^{n+1} - \frac{n}{1} p^{n-1}q + \frac{n(n-3)}{1\cdot 2} p^{n-3}q^2 - \\ - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3} p^{n-5}q^3 + \ldots - \\ - q\left(p^{n-1} - \frac{n-1}{1} p^{n-3}q + \frac{(n-1)(n-4)}{1\cdot 2} p^{n-5}q^2 - \right. \\ \left. - \frac{(n-1)(n-5)(n-6)}{1\cdot 2\cdot 3} p^{n-7}q^3 + \ldots\right) = \\ = p^{n+1} - \frac{n+1}{1} p^{n-1}q + \left\{\frac{n(n-3)}{1\cdot 2} + \frac{n-1}{1}\right\} p^{n-3}q^2 - \\ - \left\{\frac{n(n-4)(n-5)}{1\cdot 2\cdot 3} + \frac{(n-1)(n-4)}{1\cdot 2}\right\} p^{n-5}q^2 + \ldots = \\ = p^{n+1} - \frac{n+1}{1} p^{n-1}q + \frac{(n+1)(n-2)}{1\cdot 2} p^{n-3}q^2 - \\ - \frac{(n+1)(n-3)(n-4)}{1\cdot 2\cdot 3} p^{n-5}q^2 + \ldots,$$

and the theorem holds at $n + 1$.

Proposition 2° can be proved just in the same way.

Bear in mind that if x and y are the roots of a quadratic equation, then both formulas represent none other than the expression of symmetric functions of the roots of this equation in terms of its coefficients.

If we put in these formulas $x=\cos\varphi+i\sin\varphi$, $y=\cos\varphi-i\sin\varphi$, then

$$x^n+y^n=2\cos n\varphi,\quad p=x+y=2\cos\varphi,\quad q=xy=1,$$

$$\frac{x^{n+1}-y^{n+1}}{x-y}=\frac{\sin(n+1)\varphi}{\sin\varphi}.$$

Thus, we obtain an expansion of $\cos n\varphi$ and $\frac{\sin(n+1)\varphi}{\sin\varphi}$ in powers of $\cos\varphi$.

92. Put

$$x^k+y^k=S_k,\quad xy=q.$$

We have to prove that

$$S_m+C_m^1qS_{m-1}+C_{m+1}^2q^2S_{m-2}+\ldots+C_{2m-2}^{m-1}q^{m-1}S_1=1.$$

Assuming the validity of this equality, let us prove that

$$S_{m+1}+C_{m+1}^1qS_m+C_{m+2}^2q^2S_{m-1}+\ldots+C_{2m-1}^{m-1}q^{m-1}S_2+$$
$$+C_{2m}^mq^mS_1=1.$$

We may consider that x and y are the roots of the quadratic equation $\alpha^2-\alpha+q=0$.

Hence

$$S_{k+1}=S_k-qS_{k-1}$$

for any whole k.

Consequently

$$S_{m+1}=S_m-qS_{m-1},$$
$$S_m=S_{m-1}-qS_{m-2},$$
$$S_{m-1}=S_{m-2}-qS_{m-3},$$
$$\ldots\ldots\ldots\ldots$$
$$S_3=S_2-qS_1,$$
$$S_2=S_1-qS_0,$$
$$S_1=S_1.$$

Let us multiply these equalities in turn by

$$1,\quad C_{m+1}^1q,\quad C_{m+2}^2q^2,\ \ldots.\ C_{2m-1}^{m-1}q^{m-1},\quad C_{2m}^mq^m$$

and add them,

Then we obtain in the left member

$$S_{m+1}+C_{m+1}^{1}qS_m+C_{m+2}^{2}q^2S_{m-1}+\ldots+C_{2m-1}^{m-1}q^{m-1}S_2+C_{2m}^{m}q^mS_1.$$

We only have to prove that the right member is equal to 1. The right member is equal to

$$S_m+C_{m+1}^{1}qS_{m-1}+C_{m+2}^{2}q^2S_{m-2}+\ldots+C_{2m-1}^{m-1}q^{m-1}S_1+$$
$$+C_{2m}^{m}q^mS_1-qS_{m-1}-C_{m+1}^{1}q^2S_{m-2}-C_{m+2}^{2}q^3S_{m-3}-\ldots-$$
$$-C_{2m-1}^{m-1}q^mS_0.$$

or

$$S_m+(C_m^1+1)\,qS_{m-1}+(C_{m+1}^2+C_{m+1}^1)\,q^2S_{m-2}+\ldots+$$
$$+(C_{2m-2}^{m-1}+C_{2m-2}^{m-2})\,q^{m-1}S_1+C_{2m}^{m}q^mS_1-qS_{m-1}-$$
$$-C_{m+1}^{1}q^2S_{m-2}-\ldots-C_{2m-2}^{m-2}q^{m-1}S_1-C_{2m-1}^{m-1}q^mS_0=$$
$$=\{S_m+C_m^1qS_{m-1}+C_{m+1}^2q^2S_{m-2}+\ldots+C_{2m-2}^{m-1}q^{m-1}S_1\}+$$
$$+C_{2m}^{m}q^mS_1-C_{2m-1}^{m-1}q^mS_0.$$

But, by hypothesis, the braced expression is equal to 1 and $C_{2m}^{m}S_1-C_{2m-1}^{m-1}S_0=0$, since $S_1=1$, and $S_0=2$. And so, the right member is equal to 1. Furthermore, it is apparent, that at $m=1$ our equality is true. Now we can assert that it is valid for any m.

93. If $u+v=1$, then

$$u^m(1+C_m^1v+C_{m+1}^2v^2+\ldots+C_{2m-2}^{m-1}v^{m-1})+$$
$$+v^m(1+C_m^1u+C_{m+1}^2u^2+\ldots+C_{2m-2}^{m-1}u^{m-1})=1.$$

Put

$$u=\frac{x-a}{b-a},\quad v=\frac{x-b}{a-b}.$$

Then $u+v=1$. Further

$$\frac{1}{u^mv^m}=\left(\frac{1}{v^m}+C_m^1\frac{1}{v^{m-1}}+C_{m+1}^2\frac{1}{v^{m-2}}+\ldots+C_{2m-2}^{m-1}\frac{1}{v}\right)+$$
$$+\left(\frac{1}{u^m}+C_m^1\frac{1}{u^{m-1}}+C_{m+1}^2\frac{1}{u^{m-2}}+\ldots+C_{2m-2}^{m-1}\frac{1}{u}\right).$$

Hence we get our identity.

94. It is easily seen that we can always choose constants $A_1, A_2, \ldots$, so that the following identity takes place

$$(1+t)^n = 1 + t^n + A_1 t\,(1+t^{n-2}) + \\ + A_2 t^2\,(1+t^{n-4}) + \ldots .$$

Indeed, $(1+t)^n$ is a polynomial of degree n in t. Dividing it by t^n+1, we obtain a remainder (a polynomial of degree not exceeding $n-1$). We divide it by $t\,(t^{n-2}+1)$ and so on. It is clear, that the quotients thus obtained will be constants determined uniquely in the process of division. Putting $t = \frac{y}{x}$ in the identity being formed, we find

$$(x+y)^n = x^n + y^n + A_1 xy\,(x^{n-2}+y^{n-2}) + \\ + A_2 x^2 y^2\,(x^{n-4}+y^{n-4}) + \ldots .$$

To determine the coefficients $A_1, A_2, \ldots$ let us put in this identity

$$x = \cos\varphi + i\sin\varphi, \quad y = \cos\varphi - i\sin\varphi.$$

Then we have

$$(2\cos\varphi)^n = 2\cos n\varphi + 2A_1\cos(n-2)\,\varphi + \\ + 2A_2\cos(n-4)\,\varphi + \ldots .$$

Taking advantage of the known formulas for the expansion of cosine's power in terms of the cosine of multiple arcs (see Problem 10, 1° and 3°), we find the expressions for $A_1, A_2, \ldots$.

95. Let y_1 and y_2 be the roots of some quadratic equation

$$y^2 + py + q = 0.$$

Let us set up this equation, i.e. find p and q.

For this purpose we multiply the first equation by q, the second by p, the third by unity and add the results. We get

$$x_1(y_1^2 + py_1 + q) + x_2(y_2^2 + py_2 + q) = a_1 q + a_2 p + a_3 = 0,$$

since

$$y_1^2 + py_1 + q = y_2^2 + py_2 + q = 0.$$

We then multiply the second equation by q, the third by p and the fourth by unity. We get

$$x_1y_1(y_1^2+py_1+q)+x_2y_2(y_2^2+py_2+q)=a_2q+a_3p+a_4=0.$$

Thus, for determining p and q we obtain a linear system

$$a_1q+a_2p+a_3=0,$$
$$a_2q+a_3p+a_4=0.$$

On finding p and q, we determine y_1 and y_2 from the equation $y^2+py+q=0$. Knowing y_1 and y_2, we then determine x_1 and x_2, say, from the first two equations. The general system is solved in the same way. Namely, suppose $y_1, y_2, \ldots, y_n$ are the roots of a certain equation of degree n:

$$y^n+p_1y^{n-1}+p_2y^{n-2}+\ldots+p_{n-1}y+p_n=0.$$

To set up this equation multiply equation (1) by p_n, equation (2) by p_{n-1} and so on, and, finally, equation $(n+1)$ by 1 and add the results. We get

$$a_1p_n+a_2p_{n-1}+\ldots+a_{n+1}=0.$$

We then multiply equation (2) by p_n, equation (3) by p_{n-1} and so on and, finally, equation $(n+2)$ by 1 and thus obtain a second linear relationship for determining p_n, $p_{n-1}, \ldots$. Continuing this operation, we finally get n linear equations for determining the unknowns $p_1, p_2, \ldots, p_n$. If $p_1, p_2, \ldots, p_n$ are found, then to determine $y_1, y_2, \ldots, y_n$ we have to solve the equation

$$y^n+py^{n-1}+\ldots+p_{n-1}y+p_n=0.$$

To find $x_1, x_2, \ldots, x_n$ it only remains to solve a system of linear equations.

Demonstrated below is the original method of solving this system belonging to S. Ramanujan. Consider the following expression

$$\Phi(\theta)=\frac{x_1}{1-\theta y_1}+\frac{x_2}{1-\theta y_2}+\ldots+\frac{x_n}{1-\theta y_n}.$$

But

$$\frac{x_1}{1-\theta y_1}=x_1(1+\theta y_1+\theta^2y_1^2+\theta^3y_1^3+\ldots),$$

$$\frac{x_2}{1-\theta y_2}=x_2(1+\theta y_2+\theta^2y_2^2+\theta^3y_2^3+\ldots),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{x_n}{1-\theta y_n}=x_n(1+\theta y_n+\theta^2y_n^2+\theta^3y_n^3+\ldots).$$

Consequently,

$$\Phi(\theta)=(x_1+x_2+\ldots+x_n)+(x_1y_1+x_2y_2+\ldots+x_ny_n)\theta+ \\ +(x_1y_1^2+\ldots+x_ny_n^2)\theta^2+\ldots+(x_1y_1^{2n-1}+x_2y_2^{2n-1}+\ldots+ \\ +x_ny_n^{2n-1})\theta^{2n-1}+(x_1y_1^{2n}+\ldots+x_ny_n^{2n})\theta^{2n}+\ldots.$$

But by virtue of the given equations we get

$$\Phi(\theta)=a_1+a_2\theta+a_3\theta^2+\ldots+a_{2n}\theta^{2n-1}+\ldots.$$

Reducing the fractions to a common denominator, we find

$$\Phi(\theta)=\frac{A_1+A_2\theta+A_3\theta^2+\ldots+A_n\theta^{n-1}}{1+B_1\theta+\beta_2\theta^2+\ldots+B_n\theta^n}.$$

Hence

$$(a_1+a_2\theta+a_3\theta^2+\ldots+a_{2n}\theta^{2n-1}+\ldots)(1+B_1\theta+B_2\theta^2+\ldots+ \\ +B_n\theta^n)=A_1+A_2\theta+\ldots+A_n\theta^{n-1}.$$

Therefore

$$A_1=a_1,$$

$$A_2=a_2+a_1B_1,$$

$$A_3=a_3+a_2B_1+a_1B_2,$$

$$\ldots\ldots\ldots\ldots$$

$$A_n=a_n+a_{n-1}B_1+a_{n-2}B_2+\ldots+a_1B_{n-1},$$

$$0=a_{n+1}+a_nB_1+\ldots+a_1B_n,$$

$$0=a_{n+2}+a_{n+1}B_1+\ldots+a_2B_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0=a_{2n}+a_{2n-1}B_1+\ldots+a_nB_n.$$

Since the quantities $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n}$ are known, the last equations enable us to find first $B_1, B_2, \ldots, B_n$ and then $A_1, A_2, \ldots, A_n$. Knowing the quantities

A_i and B_i, we can construct a rational fraction $\Phi(\theta)$ and then expand it into partial fractions. Let, for instance, the following expansion take place

$$\Phi(\theta)=\frac{p_1}{1-q_1\theta}+\frac{p_2}{1-q_2\theta}+\frac{p_3}{1-q_3\theta}+\ldots+\frac{p_n}{1-q_n\theta}.$$

Then it is clear that

$$x_1=p_1, \quad y_1=q_1;$$
$$x_2=p_2, \quad y_2=q_2;$$
$$\cdots\cdots\cdots$$
$$x_n=p_n, \quad y_n=q_n.$$

The system is solved.

96. Eor the given case we have

$$\Phi(\theta)=\frac{2+\theta+3\theta^2+2\theta^3+\theta^4}{1-\theta-5\theta^2+\theta^3+3\theta^4-\theta^5}.$$

Expanding this fraction into partial ones, we get the following values for the unknowns

$$x=-\frac{3}{5}, \qquad p=-1,$$
$$y=\frac{18+\sqrt{5}}{10}, \quad q=\frac{3+\sqrt{5}}{2},$$
$$z=\frac{18-\sqrt{5}}{10}, \quad r=\frac{3-\sqrt{5}}{2},$$
$$u=-\frac{8+\sqrt{5}}{2\sqrt{5}}, \quad s=\frac{\sqrt{5}-1}{2},$$
$$v=\frac{8-\sqrt{5}}{2\sqrt{5}}, \qquad t=-\frac{\sqrt{5}+1}{2}.$$

97. 1° We have

$$(m, \mu)=$$
$$=\frac{(1-x)(1-x^2)\ldots(1-x^{m-\mu})(1-x^{m-\mu+1})\ldots(1-x^{m-1})(1-x^m)}{(1-x)(1-x^2)\ldots(1-x^{\mu})(1-x)(1-x^2)\ldots(1-x^{m-\mu})}.$$

Hence it is clear that

$$(m, \mu)=(m, m-\mu).$$

2° Indeed,

$$(m,\ \mu+1)=\frac{(1-x^m)(1-x^{m-1})\ldots(1-x^{m-\mu+1})(1-x^{m-\mu})}{(1-x)(1-x^2)\ldots(1-x^{\mu})(1-x^{\mu+1})}=$$

$$=\frac{(1-x^{m-1})\ldots(1-x^{m-\mu})(1-x^{m-\mu-1})}{(1-x)(1-x^2)\ldots(1-x^{\mu+1})}\cdot\frac{1-x^m}{1-x^{m-\mu-1}}.$$

Thus

$$(m,\ \mu+1)=(m-1,\ \mu+1)\frac{1-x^m}{1-x^{m-\mu-1}}=$$

$$=(m-1,\ \mu+1)\frac{1-x^{m-\mu-1}+x^{m-\mu-1}-x^m}{1-x^{m-\mu-1}}=$$

$$=(m-1,\ \mu+1)\left[1+x^{m-\mu-1}\frac{1-x^{\mu+1}}{1-x^{m-\mu-1}}\right]=$$

$$=(m-1,\ \mu+1)+x^{m-\mu-1})(m-1,\ \mu).$$

3° Using the result of 2°, we get a number of equalities

$$(m,\ \mu+1)=(m-1,\ \mu+1)+x^{m-\mu-1}(m-1,\ \mu),$$

$$(m-1,\ \mu+1)=(m-2,\ \mu+1)+x^{m-\mu-2}(m-2,\ \mu),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\mu+2,\ \mu+1)=(\mu+1,\ \mu+1)+x(\mu+1,\ \mu),$$

$$(\mu+1,\ \mu+1)=(\mu,\ \mu).$$

Adding these equalities termwise, we find

$$(m,\ \mu+1)=(\mu,\ \mu)+x(\mu+1,\ \mu)+\ldots+x^{m-\mu-1}(m-1,\ \mu).\ (*)$$

4° It is required to prove that $(m,\ \mu)$ is a polynomial. We have

$$(m,\ 1)=\frac{1-x^m}{1-x}=1+x+x^2+\ldots+x^{m-1}.$$

Thus, our proposition is true at $\mu=1$ and any m. Assuming that $(m,\ k)$ is a polynomial at $k\leqslant\mu$, by virtue of the formula (*), we may assert that $(m,\ \mu+1)$ is also a polynomial. And so, our proposition is proved by the method of mathematical induction.

5° Introduce the notation

$$f(x,\ m)=1-(m,\ 1)+(m,\ 2)-(m,\ 3)+\ldots+(-1)^m(m,\ m).$$

First let us prove that

$$f(x, m) = (1 - x^{m-1}) f(x, m-2).$$

We have

$$1 = 1,$$
$$(m, 1) = (m-1, 1) + x^{m-1},$$
$$(m, 2) = (m-1, 2) + x^{m-2}(m-1, 1),$$
$$(m, 3) = (m-1, 3) + x^{m-3}(m-1, 2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(m, m-1) = (m-1, m-1) + x(m-1, m-2),$$
$$(m, m) = (m-1, m-1).$$

Multiplying these equalities successively by ± 1 and adding the results, we get

$$f(x, m) = (1 - x^{m-1}) - (m-1, 1)(1 - x^{m-2}) + (m-1, 2) \times$$
$$\times (1 - x^{m-3}) - \ldots + (-1)^{m-2}(m-1, m-2)(1-x).$$

But

$$(1 - x^{m-2})(m-1, 1) = (1 - x^{m-1})(m-2, 1),$$
$$(1 - x^{m-3})(m-1, 2) = (1 - x^{m-1})(m-2, 2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore

$$f(x, m) = (1 - x^{m-1})\{1 - (m-2, 1) + (m-2, 2) - \ldots +$$
$$+ (-1)^{m-2}(m-2, m-2)\} = (1 - x^{m-1}) f(x, m-2).$$

Thus

$$f(x, m) = (1 - x^{m-1}) f(x, m-2),$$
$$f(x, m-2) = (1 - x^{m-3}) f(x, m-4),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

First let us assume that m is even. We get

$$f(x, m) = (1 - x^{m-1})(1 - x^{m-3})(1 - x^{m-5}) \ldots (1 - x^3) f(x, 2).$$

But

$$f(x, 2) = 1 - (2, 1) + (2, 2) = 2 - \frac{1 - x^2}{1 - x} = 1 - x.$$

Consequently, indeed,

$$f(x, m) = (1 - x^{m-1})(1 - x^{m-3}) \ldots (1 - x^3)(1 - x)$$

if m is even.

If m is odd, we have

$$f(x, m) = (1-x^{m-1})(1-x^{m-3}) \dots (1-x^2) f(x, 1).$$

But $f(x, 1) = 0$, consequently $f(x, m) = 0$ for any odd m. However, the last fact can be readily established immediately from the expression for $f(x, m)$

$$f(x, m) = 1-(m, 1)+(m, 2)-(m, 3)+\dots+(-1)^m (m, m).$$

98. 1° Put

$$1+\sum_{k=1}^{n} \frac{(1-x^n)(1-x^{n-1})\dots(1-x^{n-k+1})}{(1-x)(1-x^2)\dots(1-x^k)} x^{\frac{k(k+1)}{2}} z^k = F(n).$$

Then

$$F(n+1) = 1+\sum_{k=1}^{n+1} \frac{(1-x^{n+1})(1-x^n)\dots(1-x^{n-k+2})}{(1-x)(1-x^2)\dots(1-x^k)} x^{\frac{k(k+1)}{2}} z^k.$$

Hence

$$F(n+1)-F(n) =$$

$$= \sum_{k=1}^{n} \frac{(1-x^n)\dots(1-x^{n-k+2})}{(1-x)(1-x^2)\dots(1-x^k)} x^{\frac{k(k+1)}{2}} z^k \{1-x^{n+1}-1+$$

$$+x^{n-k+1}\}+x^{\frac{(n+1)(n+2)}{2}} z^{n+1} =$$

$$= \sum_{k=1}^{n} \frac{(1-x^n)\dots(1-x^{n-k+2})}{(1-x)(1-x^2)\dots(1-x^k)} x^{\frac{k(k+1)}{2}} z^k x^{n-k+1}(1-x^k)+$$

$$+x^{\frac{(n+1)(n+2)}{2}} z^{n+1} =$$

$$= zx^{n+1} \sum_{k=1}^{n} \frac{(1-x^n)(1-x^{n-1})\dots(1-x^{n-k+2})}{(1-x)(1-x^2)\dots(1-x^{k-1})} z^{k-1} x^{\frac{k(k-1)}{1\cdot 2}} +$$

$$+zx^{n+1} x^{\frac{n(n+1)}{2}} z^n = zx^{n+1} F(n).$$

And so

$$F(n+1)-F(n) = zx^{n+1} F(n),$$

i.e.

$$F(n+1) = (1+zx^{n+1}) F(n).$$

Therefore

$$F(n) = (1 + zx^n) F(n-1),$$
$$F(n-1) = (1 + zx^{n-1}) F(n-2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F(3) = (1 + zx^3) F(2),$$
$$F(2) = (1 + zx^2) F(1),$$
$$F(1) = 1 + xz.$$

Multiplying these equalities, we actually get

$$F(n) = (1 + xz)(1 + x^2z) \ldots (1 + x^nz).$$

2° is proved similarly.

From 1° it also follows that

$$\frac{(1-x^n)(1-x^{n-1}) \ldots (1-x^{n-k+1})}{(1-x)(1-x^2) \ldots (1-x^k)}$$

is a polynomial in x (see Problem 97).

From the same equality we can obtain Newton's binomial formula as well. Indeed

$$\frac{1-x^{n-k+1}}{1-x^k} = \frac{1+x+x^2+\ldots+x^{n-k}}{1+x+x^2+\ldots+x^{k-1}}.$$

Therefore, at $x = 1$ the last expression attains the value $\frac{n-k+1}{k}$. Consequently, we may consider that the expression

$$\frac{(1-x^n)(1-x^{n-1}) \ldots (1-x^{n-k+1})}{(1-x)(1-x^2) \ldots (1-x^k)}$$

at $x = 1$ turns into

$$\frac{n(n-1) \ldots (n-k+1)}{1 \cdot 2 \ldots k} = C_n^k$$

and formula 1° at $x = 1$ yields

$$(1+z)^n = 1 + \sum_{k=1}^{k=n} C_n^k z^k \qquad \text{(Euler).}$$

99. Readily obtained from 1° of Problem 98 at $z = -1$.

100. Put

$$C_0 + C_1(z + z^{-1}) + C_2(z^2 + z^{-2}) + \\ + \ldots + C_n(z^n + z^{-n}) = \varphi_n(z).$$

We then have

$$\varphi_n(x^2z) = \varphi_n(z)\,\frac{1+x^{2n+1}z}{xz+x^{2n}}$$

(expressing $\varphi_n(z)$ in terms of a product). Making use of $\varphi_n(z)$ expressed as a sum, we find with the aid of the last identity

$$C_k x^{2k+1}(1-x^{2n-2k}) = C_{k+1}(1-x^{2n+2k+2})$$
$$(k=0,\ 1,\ 2,\ \ldots,\ n-1).$$

Furthermore, it is obvious that $C_n = x^{n^2}$. Putting in the last relation the following values for k in succession: $n-1$, $n-2, \ldots,$ 0 and multiplying the equalities thus obtained, we find

$$C_k = \frac{(1-x^{2n+2k+2})(1-x^{2n+2k+4})\ldots(1-x^{4n})}{(1-x^2)(1-x^4)\ldots(1-x^{2n-2k})}\,x^{k^2}$$
$$(k=0,\ 1,\ \ldots,\ n-1).$$

101. 1° Put

$$\cos x + i\sin x = \varepsilon.$$

Then

$$\cos x - i\sin x = \varepsilon^{-1}.$$

Further

$$\cos lx + i\sin lx = \varepsilon^l,\ \cos lx - i\sin lx = \varepsilon^{-l}.$$

Consequently

$$\sin lx = \frac{\varepsilon^l}{2i}(1-\varepsilon^{-2l}).$$

Substituting this value of $\sin lx$ into the expression for u_k, we find

$$u_k = \frac{(1-q^{2n})(1-q^{2n-1})\ldots(1-q^{2n-k+1})}{(1-q)(1-q^2)\ldots(1-q^k)}\cdot q^{-\frac{1}{2}k(2n-k)},$$

where $q=\varepsilon^{-2}$.

The required sum is rewritten as follows

$$1-u_1+u_2-u_3+\ldots+u_{2n} = 1+$$
$$+\sum_{k=1}^{2n}(-1)^k\frac{(1-q^{2n})(1-q^{2n-1})\ldots(1-q^{2n-k+1})}{(1-q)(1-q^2)\ldots(1-q^k)}\cdot q^{-\frac{1}{2}k(2n-k)}$$

Now let us take advantage of formula 1° of Problem 98 and, replacing in it n by $2n$, put $x = q$ and $z = -q^{-n-\frac{1}{2}}$.
We then have

$$1 - u_1 + u_2 - u_3 + \ldots + u_{2n} = \prod_{k=1}^{2n} (1 - q^{k-n-\frac{1}{2}}) =$$

$$= \prod_{k=1}^{2n} (1 - \varepsilon^{2n+1-2k}) = \prod_{k=1}^{n} (1 - \varepsilon^{2k-1}) (1 - \varepsilon^{-2k+1}) =$$

$$= \prod_{k=1}^{n} 2 [1 - \cos (2k - 1) x] = 2^n \prod_{k=1}^{n} [1 - \cos (2k - 1) x].$$

2° Put (as in Problem 97)

$$\frac{(1 - q^{2n}) (1 - q^{2n-1}) \ldots (2 - q^{2n-k+1})}{(1 - q) (1 - q^2) \ldots (1 - q^k)} = (2n, k).$$

Then

$$u_k = (2n, k) q^{-\frac{1}{2} k(2n-k)},$$

where $q = \cos 2x - i \sin 2x$.
We have to compute the following sum

$$\sum_{k=0}^{2n} (-1)^k u_k^2 = \sum_{k=0}^{2n} (-1)^k (2n, k)^2 q^{-k(2n-k)},$$

where $(2n, 0) = 1$.
From Problem 98, 1° we have

$$(1 - qz) (1 - q^2 z) \ldots (1 - q^{2n} z) = \sum_{k=0}^{2n} (-1)^k (2n, k) q^{\frac{k(k+1)}{2}} z^k.$$

Put

$$(1 - qz) (1 - q^2 z) \ldots (1 - q^{2n} z) = \varphi_n (z, q).$$

We then have

$$\varphi_n (z, q) \cdot \varphi_n (-z, q) = \varphi_n (q^2, z^2).$$

Hence

$$\sum_{k=0}^{2n} (-1)^k (2n, k) q^{\frac{k(k+1)}{2}} z^k \cdot \sum_{s=0}^{2n} (2n, s) q^{\frac{s(s+1)}{2}} z^s =$$

$$= \sum_{m=0}^{2n} (-1)^m \{2n, m\} q^{m(m+1)} z^{2m},$$

where $\{2n, m\}$ is obtained from $(2n, m)$ by replacing q by q^2. Consider the coefficient of z^{2n} in both members of this equality. On the right this coefficient is equal to

$$(-1)^n \{2n, n\} q^{n(n+1)}.$$

In the left member we obtain the following expression

$$\sum_{k+s=2n} (-1)^k (2n, k)(2n, s) q^{\frac{k(k+1)}{2}+\frac{s(s+1)}{2}}.$$

But

$$(2n, 2n-k) = (2n, k),$$

therefore the last sum is equal to

$$q^{2n^2+n} \sum_{k=0}^{2n} (-1)^k (2n, k)^2 q^{k^2-2nk}.$$

And so, we have

$$q^{2n^2+n} \sum_{k=0}^{2n} (-1)^k (2n, k)^2 q^{k^2-2nk} = (-1)^n \{2n, n\} q^{n^2+n}.$$

But

$$(2n, k)^2 = u_k^2 q^{2nk-k^2},$$

hence

$$\sum_{k=0}^{2n} (-1)^k u_k^2 = (-1)^n q^{-n^2} \{2n, n\}.$$

Further

$$(2n, n) = u_n q^{\frac{1}{2}n^2}, \quad \{2n, n\} = \bar{u}_n q^{-n^2},$$

where $\bar{u}_n$ is obtained from u_n by replacing x by $2x$.
Finally,

$$\sum_{k=0}^{2n} (-1)^k u_k^2 = (-1)^n \frac{\sin(2n+2)x \sin(2n+4)x \ldots \sin 4nx}{\sin 2x \sin 4x \ldots \sin 2nx}.$$

We proceeded from

$$\sum_{k=0}^{2n} (-1)^k (2n, k)^2 q^{k^2-2nk} = (-1)^n \{2n, n\} q^{-n^2}.$$

Likewise we can obtain the following formula

$$\sum_{k=0}^{2n+1} (-1)^k (2n+1, k)^2 q^{k^2-(2n+1)k} = 0,$$

If we put $q=1$, then (n, k) turns into C_n^k and we get the formulas

$$\sum_{k=0}^{2n} (-1)^k (C_{2n}^k)^2 = (-1)^n C_{2n}^n, \quad \sum_{k=0}^{2n+1} (-1)^k (C_{2n+1}^k)^2 = 0.$$

Likewise, if we take advantage of the identity

$$\varphi_n(z, q) \cdot \varphi_n(q^n z, q) = \varphi_{2n}(z, q),$$

we get

$$\sum_{k=0}^{n} (n, k)^2 q^{k^2} = (2n, n)$$

and hence

$$\sum_{k=0}^{n} (C_n^k)^2 = C_{2n}^n$$

(see Problem 72).

SOLUTIONS TO SECTION 7

1. We have to prove that

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{a+c}.$$

However, this equality is equivalent to the following

$$\frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(a+c)}$$

or

$$\frac{b-a}{b+c} = \frac{c-b}{a+b},$$

i.e.

$$b^2 - a^2 = c^2 - b^2.$$

The last equality follows immediately from the condition of the problem.

2. If a_n is the nth term and a_m the mth term of the arithmetic progression, then we have

$$a_n = a_1 + d(n-1),$$
$$a_m = a_1 + d(m-1),$$

where d is the common difference of the progression

Hence

$$a_n - a_m = (n - m)\,d.$$

By hypothesis, we have the following equalities

$$b - c = (q - r)\,d,$$

$$c - a = (r - p)\,d,$$

$$a - b = (p - q)\,d.$$

Multiplying the first of them by a, the second by b, and the third by c, we get

$$d\,[(q - r)\,a + (r - p)\,b + (p - q)\,c] =$$

$$= a\,(b - c) + b\,(c - a) + c\,(a - b) = 0,$$

whence

$$(q - r)\,a + (r - p)\,b + (p - q)\,c = 0.$$

3. We have

$$a_p - a_q = (p - q)\,d,$$

where d is the common difference of the progression.

Since, by hypothesis,

$$a_p = q, \quad a_q = p, \text{ then } a_p - a_q = q - p,$$

therefore

$$q - p = (p - q)\,d,$$

and, consequently,

$$d = -1$$

(we assume $p - q \neq 0$).

Further

$$a_m - a_p = (m - p)\,d,$$

hence

$$a_m = a_p + (m - p)\,d = q - m + p.$$

4. We have

$$a_{p+k} = a_k + pd.$$

Let k in this equality attain successively the values: $1, 2, 3, \ldots, q$. Add termwise the q obtained equalities. We get

$$a_{p+1} + a_{p+2} + \ldots + a_{p+q} =$$

$$= a_1 + a_2 + \ldots + a_q + pqd,$$

But

$$a_{p+1} + a_{p+2} + \ldots + a_{p+q} = S_{p+q} - S_p,$$
$$a_1 + a_2 + \ldots + a_q = S_q,$$

therefore we have

$$S_{p+q} = S_p + S_q + pqd.$$

On the other hand, it is known that

$$S_p = \frac{a_1+a_p}{2} p, \quad S_q = \frac{a_1+a_q}{2} q.$$

Hence

$$\frac{2S_p}{p} - \frac{2S_q}{q} = a_p - a_q = (p-q) d$$

or

$$\frac{2(pS_p - pS_q)}{p-q} = pqd.$$

Consequently

$$S_{p+q} = S_p + S_q + \frac{2(qS_p - pS_q)}{p-q} = \frac{(p+q) S_p - (p+q) S_q}{p-q}.$$

Finally

$$S_{p+q} = \frac{p+q}{p-q}(S_p - S_q) = -(p+q).$$

5. Follows from Problem 4. However, the following method may be applied. We have

$$S_p = \frac{a_1+a_p}{2} p, \quad S_q = \frac{a_1+a_q}{2} q,$$

hence

$$\frac{a_1+a_p}{2} p = \frac{a_1+a_q}{2} q$$

or

$$[2a_1 + d(p-1)] p = [2a_1 + d(q-1)] q,$$
$$2a_1 (p-q) + d(p^2 - p - q^2 + q) = 0,$$
$$2a_1 + d(p+q-1) = 0.$$

Hence

$$a_1 + a_{p+q} = 0,$$

since

$$a_{p+q} = a_1 + d(p+q-1).$$

But

$$S_{p+q} = \frac{a_1 + a_{p+q}}{2}(p+q).$$

Consequently, indeed,

$$S_{p+q} = 0.$$

6. We have

$$S_m = \frac{a_1 + a_m}{2} m, \quad S_n = \frac{a_1 + a_n}{2} n.$$

From the given condition follows:

$$\frac{a_1 + a_m}{a_1 + a_n} = \frac{m}{n},$$

i.e.

$$\frac{2a_1 + (m-1)d}{2a_1 + (n-1)d} = \frac{m}{n}.$$

Hence

$$2a_1(n-m) + \{(m-1)n - (n-1)m\}d = 0,$$

therefore

$$a_m = a_1 + (m-1)d = \frac{d}{2} + (m-1)d = \frac{2m-1}{2}d, \quad a_n = \frac{2n-1}{2}d$$

and finally

$$\frac{a_m}{a_n} = \frac{2m-1}{2n-1}.$$

7. It is necessary to prove that at the given n and k (positive integers $k \geqslant 2$) we can find a whole s such that the following equality takes place

$$(2s+1) + (2s+3) + \ldots + (2s+2n-1) = n^k.$$

The left member is equal to

$$(2s+n)n.$$

Therefore it remains to prove that it is possible to find an integer s such that the following equality takes place

$$(2s+n)n = n^k, \quad s = \frac{n(n^{k-2}-1)}{2}.$$

But n can be either even or odd. In both cases s will be an integer, and our proposition is proved.

8. Let $a_2 = d$. Then $a_k = a_1 + d(k-1) = d(k-1)$, since, by hypothesis, $a_1 = 0$.

Consequently

$$S=\frac{2}{1}+\frac{3}{2}+\ldots+\frac{n-1}{n-2}-\left(1+\frac{1}{2}+\ldots+\frac{1}{n-3}\right)=$$

$$=\sum_{k=1}^{n-2}\frac{k+1}{k}-\sum_{k=1}^{n-2}\frac{1}{k}+\frac{1}{n-2}=\sum_{k=1}^{n-2}\left(1+\frac{1}{k}\right)-$$

$$-\sum_{k=1}^{n-2}\frac{1}{k}+\frac{1}{n-2}=\sum_{k=1}^{n-2}1+\sum_{k=1}^{n-2}\frac{1}{k}-\sum_{k=1}^{n-2}\frac{1}{k}+\frac{1}{n-2}=$$

$$=n-2+\frac{1}{n-2}=\frac{(n-2)d}{d}+\frac{d}{(n-2)d}=\frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}.$$

9. Multiplying both the numerator and denominator of each fraction on the left by the conjugate of the denominator, we get

$$S=\frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1}+\frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}=$$

$$=\frac{1}{d}\left(\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots+\sqrt{a_n}-\sqrt{a_{n-1}}\right)=$$

$$=\frac{\sqrt{a_n}-\sqrt{a_1}}{d},$$

since

$$a_2-a_1=a_3-a_2=\ldots=a_n-a_{n-1}=d.$$

Hence

$$S=\frac{\sqrt{a_n}-\sqrt{a_1}}{d}=\frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}=\frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}.$$

10. We have

$$a_1^2-a_2^2=(a_1-a_2)(a_1+a_2)=-d(a_1+a_2),$$

$$a_3^2-a_4^2=(a_3-a_4)(a_3+a_4)=-d(a_3+a_4),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{2k-1}^2-a_{2k}^2=(a_{2k-1}-a_{2k})(a_{2k-1}+a_{2k})=-d(a_{2k-1}+a_{2k}).$$

Therefore

$$S=-d(a_1+a_2+a_3+a_4+\ldots+a_{2k-1}+a_{2k})=-d\,\frac{a_1+a_{2k}}{2}\,2k.$$

But

$$a_{2k}=a_1+d(2k-1), \quad a_1-a_{2k}=-d(2k-1),$$

consequently,

$$S=-d(2k-1)\frac{a_1+a_{2k}}{2k-1}k=\frac{k}{2k-1}(a_1^2-a_{2k}^2).$$

11. 1° We have

$$S(n+2)-S(n+1)=a_{n+2},$$

$$S(n+3)-S(n)=a_{n+1}+a_{n+2}+a_{n+3}.$$

Consequently, we only have to prove that

$$a_{n+1}+a_{n+2}+a_{n+3}-3a_{n+2}=0.$$

But it is possible to prove that

$$\frac{a_r+a_s}{2}=a_{\frac{s+r}{2}}$$

(if r and s are of the same parity).

Indeed,

$$a_r+a_s=2a_1+(s-1)d+(r-1)d=$$

$$=2\left[a_1+\left(\frac{r+s}{2}-1\right)d\right]=2a_{\frac{r+s}{2}},$$

therefore

$$a_{n+1}+a_{n+3}=2a_{n+2},$$

and, consequently,

$$a_{n+1}+a_{n+2}+a_{n+3}-3a_{n+2}=0.$$

2° First of all

$$S(2n)-S(n)=a_{n+1}+\ldots+a_{2n}=\frac{a_{n+1}+a_{2n}}{2}\cdot n.$$

Now we have

$$S(3n)=a_1+a_2+\ldots+a_n+(a_{n+1}+\ldots+a_{2n})+a_{2n+1}+\ldots+$$

$$+a_{3n}=\frac{a_{n+1}+a_{2n}}{2}n+(a_n+a_{2n+1})+$$

$$+(a_{n-1}+a_{2n+2})+\ldots+(a_1+a_{3n}).$$

But since the sum of two terms of an arithmetic progression equidistant from its ends is a constant, we have

$$a_n+a_{2n+1}=a_{n-1}+a_{2n+2}=\ldots=a_1+a_{3n}=a_{n+1}+a_{2n}.$$

Therefore

$$S(3n)=\frac{a_{n+1}+a_{2n}}{2}n+(a_{n+1}+a_{2n})\cdot n=3\,\frac{a_{n+1}+a_{2n}}{2}n=$$
$$=3\,(S(2n)-S(n)).$$

12. According to our notation we have

$$S_k=a_{(k-1)n+1}+a_{(k-1)n+2}+\ldots+a_{kn},$$
$$S_{k+1}=a_{kn+1}+a_{kn+2}+\ldots+a_{(k+1)n}.$$

Consider the difference

$$S_{k+1}-S_k.$$

We have

$$S_{k+1}-S_k=[a_{kn+n}-a_{kn}]+\ldots+[a_{kn+2}-a_{(k-1)n+2}]+$$
$$+[a_{kn+1}-a_{(k-1)n+1}].$$

But since

$$a_m-a_l=(m-l)\,d,$$

we have

$$S_{k+1}-S_k=nd+\ldots+nd+nd=n^2d.$$

13. We have

$$b-a=d\,(q-p),\quad c-b=d\,(r-q),\quad c-a=d\,(r-p);$$

on the other hand,

$$a=u_1\omega^{p-1},\quad b=u_1\omega^{q-1},\quad c=u_1\omega^{r-1},$$

where u_1 is the first term of the geometric progression, and ω is its ratio.

Therefore

$$a^{b-c}\cdot b^{c-a}\cdot c^{a-b}=a^{d(q-r)}\cdot b^{d(r-p)}\cdot c^{d(p-q)}=$$
$$=u_1^{d(q-r)+d(r-p)+d(p-q)}\cdot\omega^{d\{(q-r)(p-1)+(r-p)(q-1)+(p-q)(r-1)\}}.$$

But it is easily seen that

$$d\,(q-r)+d\,(r-p)+d\,(p-q)=0,$$
$$(q-r)\,(p-1)+(r-p)\,(q-1)+(p-q)\,(r-1)=0.$$

And so

$$a^{b-c}\cdot b^{c-a}\cdot c^{a-b}=1.$$

14. We have

$$1+x+x^2+\ldots+x^n=\frac{x^{n+1}-1}{x-1}.$$

Consequently

$$(1+x+x^2+\ldots+x^n)^2-x^n=\left(\frac{x^{n+1}-1}{x-1}\right)^2-x^n=$$

$$=\frac{(x^{n+1}-1)^2-x^n(x-1)^2}{(x-1)^2}=\frac{x^{2n+2}-2x^{n+1}+1-x^{n+2}+2x^{n+1}-x^n}{(x-1)^2}=$$

$$=\frac{(x^n-1)(x^{n+2}-1)}{(x-1)(x-1)}=(1+x+x^2+\ldots+x^{n-1})\times$$

$$\times(1+x+x^2+\ldots+x^{n+1}).$$

15. Let the considered geometric progression be

$$u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{2n}, u_{2n+1}, \ldots, u_{3n}.$$

Hence

$$S_{3n}-S_{2n}=u_{2n+1}+\ldots+u_{3n},\quad S_{2n}-S_n=u_{n+1}+\ldots+u_{2n}.$$

But

$$u_k=u_1q^{k-1},\qquad u_s=u_1q^{s-1}.$$

Therefore

$$u_k=u_s\cdot q^{k-s},\qquad u_{2n+k}=u_kq^{2n},$$

consequently,

$$S_{3n}-S_{2n}=u_{2n+1}+\ldots+u_{3n}=q^{2n}(u_1+u_2+\ldots+u_n)=q^{2n}S_n,$$

$$S_{2n}-S_n=u_{n+1}+\ldots+u_{2n}=q^n(u_1+u_2+\ldots+u_n)=q^nS_n.$$

Therefore

$$S_n(S_{3n}-S_{2n})=q^{2n}S_n^2,\qquad (S_{2n}-S_n)^2=q^{2n}S_n^2,$$

and the problem is solved.

16. Using the formula for the sum of terms of the geometric progression, we get

$$S=\frac{a_nq-a_1}{q-1},\qquad S'=\frac{\frac{1}{a_n}\frac{1}{q}-\frac{1}{a_1}}{\frac{1}{q}-1}=\frac{a_nq-a_1}{q-1}\cdot\frac{1}{a_na_1}.$$

Consequently

$$\frac{S}{S'}=a_na_1.$$

But, on the other hand,

$$P^2=(a_1a_2\ldots a_n)^2=(a_1a_n)^n,$$

hence

$$P=\left(\frac{S}{S'}\right)^{\frac{n}{2}}.$$

17. Let us consider Lagrange's identity mentioned in Sec. 1 (see Problem 5)

$$(x_1^2+x_2^2+\ldots+x_{n-1}^2)(y_1^2+y_2^2+\ldots+y_{n-1}^2)-$$
$$-(x_1y_1+x_2y_2+\ldots+x_{n-1}y_{n-1})^2=(x_1y_2-x_2y_1)^2+$$
$$+(x_1y_3-x_3y_1)^2+\ldots+(x_{n-2}y_{n-1}-y_{n-2}x_{n-1})^2.$$

Put

$$x_1=a_1,\quad x_2=a_2,\quad \ldots,\quad x_{n-1}=a_{n-1};$$
$$y_1=a_2,\quad y_2=a_3,\quad \ldots,\quad y_{n-1}=a_n.$$

We then have

$$(a_1^2+a_2^2+\ldots+a_{n-1}^2)(a_2^2+a_3^2+\ldots+a_n^2)-$$
$$-(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)^2=(a_1a_3-a_2^2)^2+$$
$$+(a_1a_4-a_3a_2)^2+\ldots+(a_{n-2}a_n-a_{n-1}^2)^2. \qquad (*)$$

The bracketed expressions on the right have the following structure

$$a_ka_s-a_{k'}a_{s'},$$

and $k+s=k'+s'$. It is evident that if $a_1, a_2, \ldots, a_n$ form a geometric progression, then (provided $k+s=k'+s'$)

$$a_ka_s-a_{k'}a_{s'}=0.$$

Indeed

$$a_k=a_1q^{k-1},\qquad a_s=a_1q^{s-1},$$
$$a_{k'}=a_1q^{k'-1},\quad a_{s'}=a_1q^{s'-1}.$$

Therefore

$$a_ka_s=a_1^2q^{k+s-2}$$

and

$$a_{k'}a_{s'}=a_1^2q^{k'+s'-2},$$
$$a_ka_s=a_{k'}a_{s'}.$$

Thus, if $a_1, a_2, \ldots, a_n$ form a geometric progression, then all the bracketed expressions in the right member of the equality (*) are equal to zero, and the following relation takes place

$$(a_1^2+a_2^2+\dots+a_{n-1}^2)(a_2^2+a_3^2+\dots+a_n^2)=$$
$$=(a_1a_2+a_2a_3+\dots+a_{n-1}a_n)^2.$$

Now let us assume that this relation takes place. It is required to prove that the numbers $a_1, a_2, \dots, a_n$ form a geometric progression. In this case all the bracketed expressions in the right member of the equality (*) are equal to zero. But among these expressions there is the following one

$$(a_1a_k - a_2a_{k-1})^2 \quad (k=3, 4, \dots, n).$$

Therefore we have

$$\frac{a_k}{a_{k-1}}=\frac{a_2}{a_1} \quad (k=3, 4, \dots, n),$$

i.e. the numbers $a_1, a_2, \dots, a_n$ really form a geometric progression.

18. 1° It is known that

$$S_m=\frac{a_mq-a_1}{q-1}.$$

Let us make up the required sum. We have

$$S_1+S_2+\dots+S_n=\frac{a_1q-a_1}{q-1}+\frac{a_2q-a_1}{q-1}+\dots+\frac{a_nq-a_1}{q-1}=$$
$$=\frac{(a_1+a_2+\dots+a_n)q}{q-1}-\frac{a_1n}{q-1}=\frac{(a_nq-a_1)q}{(q-1)^2}-\frac{a_1n}{q-1}$$

2°

$$\frac{1}{a_1^2-a_2^2}+\dots+\frac{1}{a_{n-1}^2-a_n^2}=\frac{1}{1-q^2}\left\{\frac{1}{a_1^2}+\frac{1}{a_2^2}+\dots+\frac{1}{a_{n-1}^2}\right\}=$$
$$=\frac{1}{1-q^2}\,\frac{\frac{1}{a_{n-1}^2}\cdot\frac{1}{q^2}-\frac{1}{a_1^2}}{\frac{1}{q^2}-1}=q^2\,\frac{\left(\frac{1}{a_n^2}-\frac{1}{a_1^2}\right)}{(1-q^2)^2}$$

3°

$$\frac{1}{a_1^k+a_2^k}+\dots+\frac{1}{a_{n-1}^k+a_n^k}=\frac{1}{1+q^k}\,\frac{q^k\left(\frac{1}{a_n^k}-\frac{1}{a_1^k}\right)}{1-q^k}=$$
$$=\frac{q^k}{1-q^{2k}}\left(\frac{1}{a_n^k}-\frac{1}{a_1^k}\right).$$

19. Let the given progression be $a_1, a_2, \dots, a_n$. Let $a_{\bar{k}}$ designate the kth term from the end of the progression. Then

$$a_{\bar{k}}=a_n-(k-1)d, \quad a_k=a_1+(k-1)d.$$

Consider the product $a_k a_{\bar{k}}$. We have

$$a_k a_{\bar{k}} = a_1 a_n - (k-1)^2 d^2 + (k-1) d (a_n - a_1) =$$
$$= a_1 a_n - (k-1)^2 d^2 + (k-1)(n-1) d^2.$$

And so

$$a_k a_{\bar{k}} = a_1 a_n + d^2 \{(k-1)(n-1) - (k-1)^2\}.$$

It only remains to prove that the expression

$$P_n = (k-1)(n-1) - (k-1)^2$$

increases with an increase in n from 1 to $\frac{n}{2}$ or $\frac{n+1}{2}$.

We have

$$P_k = (k-1)(n-k), \quad P_{k+1} = k(n-k-1).$$

Hence

$$P_{k+1} - P_k = n - 2k.$$

Consequently, $P_{k+1} > P_k$ if $n - 2k > 0$, i.e. if $k < \frac{n}{2}$, and our proposition is proved.

20. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression, and $u_1, u_2, \ldots, u_n$ a geometric progression. By hypothesis, $a_1 = u_1$, $a_n = u_n$. Let the ratio of the progression be equal to q. Then

$$u_n = u_1 q^{n-1} = a_n.$$

Put

$$a_1 + a_2 + \ldots + a_n = s_n, \quad u_1 + u_2 + \ldots + u_n = \sigma_n.$$

Prove that

$$s_n \geqslant \sigma_n.$$

We have

$$s_n = \frac{a_1 + a_n}{2} \cdot n = \frac{a_1 + a_1 q^{n-1}}{2} n = a_1 \frac{1 + q^{n-1}}{2} n,$$

$$\sigma_n = \frac{u_n q - u_1}{q-1} = a_1 \frac{q^n - 1}{q-1}.$$

Since, by hypothesis, $a_1 > 0$, it only remains to prove that

$$\frac{q^n - 1}{q-1} \leqslant \frac{1 + q^{n-1}}{2} n.$$

Let us write the left member of the supposed inequality in the following way

$$\frac{q^n-1}{q-1}=1+q+q^2+\ldots+q^{n-3}+q^{n-2}+q^{n-1}=$$
$$=\frac{1}{2}\{(1+q^{n-1})+(q+q^{n-2})+\ldots+(q^k+q^{n-k-1})+\ldots+$$
$$+(q^{n-1}+1)\}.$$

Let us prove that

$$q^k+q^{n-k-1}\leqslant 1+q^{n-1}.$$

Indeed

$$q^k+q^{n-k-1}-1-q^{n-1}=(q^k-1)+q^{n-k-1}(1-q^k)=$$
$$=(q^k-1)(1-q^{n-k-1})\leqslant 0,$$

since if $q>1$, then $q^k-1\geqslant 0$, $1-q^{n-k-1}\leqslant 0$, and if $q<1$, then $q^k-1\leqslant 0$, $1-q^{n-k-1}\geqslant 0$. At $q=1$ it is clear that the product contained in the left member of our inequality is equal to zero. And so, indeed,

$$q^k+q^{n-k-1}\leqslant 1+q^{n-1}.$$

The braced expression contains n bracketed expressions each of which does not exceed $1+q^{n-1}$. Therefore

$$\frac{q^n-1}{q-1}\leqslant n\,\frac{1+q^{n-1}}{2},$$

i.e.

$$\sigma_n\leqslant s_n,$$

which solves the problem.

21. Let the first common term of the progressions be a, and the second b. Then the nth term of the arithmetic progression will be equal to

$$a+(b-a)(n-1),$$

and the corresponding term of the geometric progression has the form

$$a\left(\frac{b}{a}\right)^{n-1}.$$

And so, we have to prove that

$$a+(b-a)(n-1)\leqslant a\left(\frac{b}{a}\right)^{n-1},$$

in other words, that

$$a+(b-a)(n-1)-a\left(\frac{b}{a}\right)^{n-1}\leqslant 0,$$

or

$$a\left\{\left(\frac{b}{a}-1\right)(n-1)-\left[\left(\frac{b}{a}\right)^{n-1}-1\right]\right\}\leqslant 0.$$

Let us rewrite the left member of this inequality as follows

$$a\left(\frac{b}{a}-1\right)\left\{(n-1)-\left[\left(\frac{b}{a}\right)^{n-2}+\left(\frac{b}{a}\right)^{n-3}+\ldots+\left(\frac{b}{a}\right)+1\right]\right\}.$$

Considering separately the three cases: $\frac{b}{a}>1$, $\frac{b}{a}<1$, $\frac{b}{a}=1$, we easily prove the validity of our inequality.

22. We have to compute

$$S_n=1\cdot x+2x^2+3x^3+\ldots+nx^n.$$

Multiplying both members of this equality by x, we have

$$S_nx=1\cdot x^2+2x^3+3x^4+\ldots+(n-1)x^n+nx^{n+1}.$$

It is evident that the right member is equal to

$$S_n-x-x^2-x^3-\ldots-x^n+nx^{n+1}.$$

Thus, we have the identity

$$S_nx=S_n+nx^{n+1}-x(1+x+x^2+\ldots+x^{n-1}),$$

$$S_n(x-1)=nx^{n+1}-x\frac{x^n-1}{x-1},$$

$$S_n(x-1)^2=x\{nx^{n+1}+1-(n+1)x^n\}.$$

And, finally, we have

$$S_n=\frac{x}{(x-1)^2}\{nx^{n+1}-(n+1)x^n+1\}.$$

23. We have

$$s=\sum_{k=1}^{n}a_ku_k.$$

Let us multiply both members of this equality by q (where q is the ratio of the geometric progression). We obtain

$$sq = \sum_{k=1}^{n} a_k u_{k+1}$$

(since $u_k q = u_{k+1}$).

Subtract s from both members of the last equality. We have

$$sq - s = \sum_{k=1}^{n} a_k u_{k+1} - \sum_{k=1}^{n} a_k u_k.$$

Transform the right member as follows

$$\sum_{=2}^{n+1} a_{k-1}u_k - \sum_{k=2}^{n+1} a_k u_k - a_1u_1 + a_{n+1}u_{n+1} =$$

$$= - \sum_{k=2}^{n+1} (a_k - a_{k-1}) u_k - a_1u_1 + a_{n+1}u_{n+1} =$$

$$= - \sum_{k=2}^{n+1} du_k + a_{n+1}u_{n+1} - a_1u_1,$$

where d is the common difference of the arithmetic progression.

Thus

$$s(q-1) = -d \sum_{k=2}^{n+1} u_k + a_{n+1}u_{n+1} - a_1u_1,$$

$$s(q-1) = a_{n+1}u_{n+1} - a_1u_1 - d\frac{u_{n+1}q - u_2}{q-1}.$$

Finally

$$s = \frac{a_{n+1}u_{n+1} - a_1u_1}{q-1} - d\frac{u_{n+1}q - u_2}{(q-1)^2}.$$

24. The required sum can be rewritten in the following way

$$x^2 + x^4 + \ldots + x^{2n} + \frac{1}{x^2} + \frac{1}{x^4} + \ldots + \frac{1}{x^{2n}} + 2n.$$

Summing each of the geometric progressions separately and joining the partial sums thus obtained, we have

$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \ldots + \left(x^n + \frac{1}{x^n}\right)^2 =$$

$$= \frac{(x^{2n+2}+1)(x^{2n}-1)}{(x^2-1)x^{2n}} + 2n.$$

25. The sum S_1 is readily computed by the formula for an arithmetic progression. Let us now compute S_2. Consider the following identity

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1.$$

Putting here in succession $x = 1, 2, 3, \ldots, n$ and summing up the obtained equalities termwise, we have

$$\sum_{x=1}^{n} (x+1)^3 - \sum_{x=1}^{n} x^3 = 3\sum_{x=1}^{n} x^2 + 3\sum_{x=1}^{n} x + n.$$

Or

$$\{2^3 + 3^3 + \ldots + n^3 + (n+1)^3\} - \{1^3 + 2^3 + \ldots + n^3\} = \\ = 3S_2 + 3S_1 + n.$$

And so $3S_2 + 3S_1 + n = (n+1)^3 - 1$. But

$$S_1 = \frac{n(n+1)}{2}.$$

Now we find easily

$$S_2 = \frac{n(n+1)(2n+1)}{6}.$$

The formula for S_3 is deduced in a similar way. We only have to consider the identity

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$$

and make use of the expressions for S_1 and S_2 found before.

26. We have identically

$$(x+1)^{k+1} - x^{k+1} = (k+1)x^k + \frac{(k+1)k}{1\cdot 2}x^{k-1} + \\ + \frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}x^{k-2} + \ldots + (k+1)x + 1.$$

Putting here successively $x = 1, 2, 3, \ldots, n$, and summing up, we get the required formula.

27. Consider the following square table:

1^k	2^k	3^k	4^k	$\dots n^k$
1^k	2^k	3^k	4^k	$\dots n^k$
1^k	2^k	3^k	4^k	$\dots n^k$
. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .
1^k	2^k	3^k	4^k	$\dots n^k$

The sum of terms of each line is equal to $1^k + 2^k + \dots + n^k = S_k(n)$. Thus, the sum of all the terms of the table will be $nS_k(n)$.

On the other hand, summing along the broken lines, we get the following expression for the sum of all the terms of the table

$$1^k + (1^k + 2\cdot 2^k) + (1^k + 2^k + 3\cdot 3^k) + (1^k + 2^k + 3^k + 4\cdot 4^k) + \dots + (1^k + 2^k + 3^k + \dots + (n-1)^k + n\cdot n^k) =$$

$$= 1 + [S_k(1) + 2^{k+1}] + [S_k(2) + 3^{k+1}] + [S_k(3) + 4^{k+1}] + \dots + [S_k(n-1) + n^{k+1}] =$$

$$= S_k(1) + S_k(2) + \dots + S_k(n-1) + (1^{k+1} + 2^{k+1} + 3^{k+1} + \dots + n^{k+1}).$$

And so

$$nS_k(n) = S_{k+1}(n) + S_k(n-1) + S_k(n-2) + \dots + S_k(2) + S_k(1).$$

28. Both 1° and 2° are readily obtained from the formula of Problem 26. Let us rewrite it as

$$S_k = -\frac{k}{2} S_{k-1} - \frac{k(k-1)}{1\cdot 2\cdot 3} S_{k-2} - \dots - S_1 - \frac{S_0}{k+1} + \frac{(n+1)^{k+1} - 1}{k+1}. \quad (*)$$

At $k = 1$ $S_1 = 1 + 2 + 3 + \dots + n = \frac{n^2+n}{2} = \frac{1}{2} n^2 + \frac{1}{2} n.$

Thus, both propositions (1° and 2°) are valid at $k = 1$. Suppose they hold true for any value of the subscript less than k and let us prove that they are also valid at the subscript equal to k. Since, by supposition, S_{k-1} is a polynomial in n of degree k, S_{k-2} a polynomial of degree $k - 1$, and so on, it is easily seen from the formula (*) that S_k is indeed a polynomial of degree $k+1$. Further, since $S_{k-1}, S_{k-2}, \ldots, S_0$ do not contain the term independent of n, it follows that S_k also does not contain such a term $\left(\frac{(n+1)^{k+1}-1}{k+1}\right.$, when expanded in powers of n, will not contain a constant $\left.\text{term}\right)$. As is evident from the same formula (*), the coefficient of the term of the highest power in the expansion of S_k in powers of n will be $\frac{1}{k+1}$. It only remains to prove that the coefficient of the second term, i.e. B, is equal to $\frac{1}{2}$. In the expansion (*) there exist only two terms containing n^k. One of them is contained in $-\frac{k}{2}S_{k-1}$, and the other in $\frac{(n+1)^{k+1}-1}{k+1}$. From what has been proved we have

$$-\frac{k}{2}S_{k-1}=-\frac{k}{2}\left\{\frac{1}{k}n^k+\ldots\right\}=-\frac{1}{2}n^k+\ldots .$$

Further

$$\frac{(n+1)^{k+1}-1}{k+1}=\frac{1}{k+1}n^{k+1}+n^k+\ldots .$$

Hence, it is obvious that

$$B=\frac{1}{2}.$$

As to the structure of the rest of the coefficients ($C, \ldots, L$), we may assert the following: the coefficient of n^{k+1-l} will be equal to

$$C_{k+1}^{l}\frac{A}{k+1},$$

where A is independent of k. This proposition is proved using the method of induction with the aid of the formula (*).

29. S_4 can be computed using, for instance, the formula from Problem 26.

However, we may also proceed in the following way. From the result of the previous problem it follows that

$$S_4=\frac{1}{5}n^5+\frac{1}{2}n^4+Cn^3+Dn^2+En.$$

It only remains to determine C, D and E. Since the last equality is an identity, it is valid for all values of n. Putting here in succession $n=1, 2$, and 3, we get a system of equations in three unknowns C, D and E. Namely, we have

$$C+D+E=\frac{3}{10},\ 8C+4D+2E=\frac{13}{5},\ 27+C+9D+3E=\frac{89}{10}.$$

Hence

$$C=\frac{1}{3},\ D=0,\ E=-\frac{1}{30}.$$

It only remains to factor the expression

$$\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$$

and the required result will be found.

The remaining three formulas are obtained similarly.

30. The validity of the identities is established by a direct check, using the expressions for S_n obtained before.

31. Put $k = 1$. We have

$$(B+1)^2 - B^2 = 2,$$

or

$$B_2 + 2B_1 + 1 - B_2 = 2.$$

Consequently, $B_1 = \frac{1}{2}$.

Then put $k = 2$. We get

$$(B+1)^3 - B^3 = 3,$$

i.e.

$$B_3+3B_2+3B_1+1-B_3=3, \quad \text{i.e.} \quad B_2=\frac{1}{6}.$$

Proceeding in the same way, we get the following table

$B_1=\frac{1}{2}$,	$B_6=\frac{1}{42}$,	$B_{11}=0$,	$B_{16}=-\frac{3617}{510}$,
$B_2=\frac{1}{6}$,	$B_7=0$,	$B_{12}=-\frac{691}{2730}$,	$B_{17}=0$;
$B_3=0$,	$B_8=-\frac{1}{30}$,	$B_{13}=0$,	$B_{18}=\frac{43867}{798}$,
$B_4=-\frac{1}{30}$,	$B_9=0$,	$B_{14}=\frac{7}{6}$,	$B_{19}=0$.
$B_5=0$,	$B_{10}=\frac{5}{66}$,	$B_{15}=0$,	

Knowing this table, we may easily solve Problem 29, i.e. arrange S_4, S_5, S_6 and S_7 according to powers of n. These numbers play quite an important role in many fields of mathematics and possess a number of interesting properties. They are called Bernoulli's numbers (J. Bernoulli, *Ars Conjectandi*). We can show that for odd k's exceeding unity B_k will be equal to zero. And Bernoulli's numbers with an even subscript will increase rather fast. Let us consider the value of B_{196}. If we put $B_{196} = -\frac{Z}{N}$, then it turns out that

$$N = 171390,$$

$$\begin{aligned} Z = {} & 62753\ 13511\ 04611\ 93672\ 55310\ 66998 \\ & 93713\ 60315\ 30541\ 53311\ 89530\ 55906 \\ & 39107\ 01782\ 46402\ 41378\ 48048\ 46255 \\ & 54578\ 57614\ 21158\ 35788\ 96086\ 55345 \\ & 32214\ 56098\ 29255\ 49798\ 68376\ 27052 \\ & 31316\ 61171\ 66687\ 49347\ 22145\ 80056 \\ & 71217\ 06735\ 79434\ 16524\ 98443\ 87718 \\ & 31115 \end{aligned}$$

Thus, the numerator of this number contains 215 digits (D. H. Lehmer, 1935).

Let us now prove relationship 2°.

On the basis of the results obtained in Problem 28 we may put

$$(k+1)(1^k+2^k+3^k+\ldots+n^k) = \\ = n^{k+1} + \frac{k+1}{2} n^k + Cn^{k-1} + Dn^{k-2} + \ldots + Ln,$$

where $C, D, \ldots, L$ are independent of n, but undoubtedly depend on k. Put

$$(k+1)(1^k+2^k+3^k+\ldots+n^k) = n^{k+1} + C_{k+1}^1\alpha_1 n^k + \\ + C_{k+1}^2\alpha_2 n^{k-1} + \ldots + C_{k+1}^{k-1}\alpha_{k-1}n^2 + C_{k+1}^k\alpha_k n.$$

We may then write the following symbolic equality

$$(k+1)(1^k+2^k+\ldots+n^k) = (n+\alpha)^{k+1} - \alpha^{k+1}.$$

On removing the brackets in the right member by replacing α^s by α_s ($s=0, 1, 2, \ldots$), we pass over from the symbolic equality to an ordinary one.

Since this equality is an identity with respect to n, we may put in it $n+1$ instead of n and obtain

$$(k+1)[1^k+2^k+\ldots+(n+1)^k] = (n+1+\alpha)^{k+1} - \alpha^{k+1}$$

Subtracting from the last equality the preceding one, we find

$$(k+1)(n+1)^k = (n+1+\alpha)^{k+1} - (n+\alpha)^{k+1}.$$

Putting here $n=0$, we have

$$(\alpha+1)^{k+1} - \alpha^{k+1} = k+1.$$

Besides, it should be remembered (see the solution of Problem 28) that α's are independent of k and that $\alpha_1 = \frac{1}{2}$.

And so, the numbers α_k and B_k are determined by one and the same relation, and $\alpha_1 = B_1$. Therefore

$$\alpha_k = B_k$$

for any k.

32. Let d be the common difference of our progression. Then

$$x_k = x_1 + d(k-1).$$

From the first equality we have

$$\frac{x_1+x_n}{2} n = a, \quad nx_1 + d\frac{n(n-1)}{1\cdot 2} = a. \tag{*}$$

On the other hand,

$$x_k^2 = x_1^2 + 2x_1 d(k-1) + d^2(k-1)^2.$$

Therefore, from the second relation we get

$$\sum_{k=1}^{n} x_k^2 = nx_1^2 + 2x_1 d \sum_{k=1}^{n}(k-1) + d^2 \sum_{k=1}^{n}(k-1)^2 = b^2.$$

Hence

$$nx_1^2 + 2x_1 d\frac{n(n-1)}{1\cdot 2} + d^2\frac{(n-1)n(2n-1)}{6} = b^2 \tag{1}$$

(see Problem 25).

Squaring both members of the equality (*) and dividing by n, we find

$$nx_1^2 + 2x_1 d\frac{n(n-1)}{1\cdot 2} + d^2\frac{n(n-1)^2}{4} = \frac{a^2}{n}. \tag{2}$$

Subtracting (2) from (1), we get

$$\frac{d^2 n(n^2-1)}{12} = \frac{b^2 n - a^2}{n}.$$

Consequently

$$d = \pm\frac{2\sqrt{3(b^2n-a^2)}}{n\sqrt{n^2-1}}.$$

Substituting d into the equality (*), we find x_1, and, consequently, we can construct the whole arithmetic progression.

33. 1° Put $s=\sum_{k=1}^{n} k^2x^{k-1}$. Hence $x\cdot s=\sum_{k=1}^{n} k^2x^k$.

Subtracting the first equality from the second, we find

$$s(x-1)=\sum_{k=2}^{n+1}(k-1)^2x^{k-1}-\sum_{k=1}^{n}k^2x^{k-1}.$$

Consequently

$$s(x-1)=\sum_{k=1}^{n}(k-1)^2x^{k-1}+n^2x^n-\sum_{k=1}^{n}k^2x^{k-1},$$

$$s(x-1)=n^2x^n-\sum_{k=1}^{n}(2k-1)\,x^{k-1}=n^2x^n-2\sum_{k=1}^{n}kx^{k-1}+$$

$$+\sum_{k=1}^{n}x^{k-1}=n^2x^n-2\frac{1}{(x-1)^2}\{nx^{n+1}-(n+1)\,x^n+1\}+\frac{x^n-1}{x-1}$$

(see Problem 22).

Finally

$$1+4x+9x^2+\ldots+n^2x^{n-1}=$$

$$=\frac{n^2x^n(x-1)^2-2nx^n(x-1)+(x^n-1)(x+1)}{(x-1)^3}.$$

2° Proceed as in the previous case. Put

$$s=1^3+2^3x+3^3x^2+\ldots+n^3x^{n-1}=\sum_{k=1}^{n}k^3x^{k-1}.$$

Make up the difference

$$sx-s=n^3x^n-3\sum_{k=1}^{n}k^2x^{k-1}+3\sum_{k=1}^{n}kx^{k-1}-\sum_{k=1}^{n}x^{k-1}.$$

Substituting the expressions obtained before for the sums on the right, we have

$$s(x-1)=n^3x^n-3\frac{n^2x^n(x-1)^2-2nx^n(x-1)+(x^n-1)(x+1)}{(x-1)^3}+$$

$$+3\frac{nx^{n+1}-(n+1)\,x^n+1}{(x-1)^2}-\frac{x^n-1}{x-1}.$$

Finally

$$s(x-1)^4 = n^3x^n(x-1)^3 - 3n^2x^n(x-1)^2 + \\ + 3nx^n(x^2-1) - (x^n-1)(x^2+4x+1).$$

34. To determine the required sums first compute the following sum

$$1+3x+5x^2+\ldots+(2n-1)x^{n-1} = \sum_{k=1}^{n}(2k-1)x^{k-1} = \\ = 2\sum_{k=1}^{n} kx^{k-1} - \sum_{k=1}^{n} x^{k-1} = \frac{2nx^n(x-1)-(x+1)(x^n-1)}{(x-1)^2}.$$

For computing the first of the sums put in the deduced formula $x=\frac{1}{2}$. We then have

$$1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots+\frac{2n-1}{2^{n-1}} = \frac{1}{2^{n-1}}\{3(2^n-1)-2n\}.$$

And putting $x=-\frac{1}{2}$, we find

$$1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots+(-1)^{n-1}\frac{2n-1}{2^{n-1}} = \frac{2^n+(-1)^{n+1}(6n+1)}{9\cdot 2^{n-1}}.$$

35. 1° First assume that n is even. Put $n=2m$. Then

$$1-2+3-4+\ldots+(-1)^{n-1}n = \\ = 1-2+3-4+\ldots+(2m-1)-2m = (1+3+\ldots+ \\ + 2m-1) - (2+4+\ldots+2m) = -m = -\frac{n}{2}.$$

Now let n be odd and put $n=2m-1$. Then our sum takes the form

$$[1-2+3-4+\ldots-(2m-2)]+(2m-1) = \\ = -(m-1)+2m-1 = m = \frac{n+1}{2}$$

Thus, if we put

$$1-2+3-4+\ldots+(-1)^{n-1}n = S,$$

then

$$S = -\frac{n}{2} \text{ if } n \text{ is even, } S = \frac{n+1}{2} \text{ if } n \text{ is odd.}$$

However, this result can be obtained in a simpler way. Indeed, if n is even, we have

$$S = [1-2]+[3-4]+[5-6]+\ldots+[(2m-1)-2m] = \\ = -1\cdot m = -m = -\frac{n}{2}.$$

Hence we also get the result for odd n.

2° First assume that n is even and put $n=2m$. We have

$$1^2-2^2+3^2-\ldots+(-1)^{n-1}n^2=(1^2-2^2)+$$
$$+(3^2-4^2)+\ldots+[(2m-1)^2-(2m)^2]=-(1+2)-$$
$$-(3+4)-\ldots-(2m-1+2m)=-[1+2+3+4+\ldots+$$
$$+2m-1+2m]=-\frac{(2m+1)\,2m}{2}=-\frac{n(n+1)}{2}.$$

Thus, if n is even, then

$$1^2-2^2+3^2-\ldots+(-1)^{n-1}n^2=-\frac{n(n+1)}{1\cdot 2}.$$

If $n=2m+1$ is odd, then

$$1^2-2^2+3^2-\ldots+(-1)^{n-1}n^2=1^2-2^2+3^2-4^2-\ldots-$$
$$-(2m)^2+(2m+1)^2=\frac{-2m(2m+1)}{2}+(2m+1)^2=$$
$$=n^2-\frac{n(n-1)}{2}=\frac{n(n+1)}{1\cdot 2}.$$

3° The required sum is equal to $-8n^2$. The result is obtained as in the previous case.

4° Rewrite the required sum as

$$\sum_{k=1}^{n}(k^3+k^2)=\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k^2=\frac{n(n+1)(3n^2+7n+2)}{12}$$

(see Problem 25).

36. The considered sum may be rewritten as

$$\frac{10-1}{9}+\frac{10^2-1}{9}+\frac{10^3-1}{9}+\ldots+\frac{10^n-1}{9},$$

wherefrom we easily find its value

$$\frac{1}{9}\left\{10\,\frac{10^n-1}{9}-n\right\}.$$

37. Consider the first bracketed expression on the right and rewrite it in the following way

$$2x^{2n+1}-2x^{2n-1}y^2+2x^{2n-3}y^4-\ldots\pm 2xy^{2n}-x^{2n+1}=$$
$$=2x\,\frac{x^{2n+2}+y^{2n+2}}{x^2+y^2}-x^{2n+1}.$$

The second bracketed expression arises from the first one as a result of permutation of the letters x and y, therefore it is equal to $2y\frac{x^{2n+2}+y^{2n+2}}{x^2+y^2}-y^{2n+1}$. Squaring both obtained expressions and adding the results, we easily prove the validity of the identity.

38. The required product is equal to

$$(1\cdot a+1\cdot a^2+\ldots+1\cdot a^{n-1})+(a\cdot a^2+\ldots+aa^{n-1})+$$
$$+(a^2a^3+\ldots+a^2\cdot a^{n-1})+\ldots+a^{n-2}\cdot a^{n-1}=$$
$$=a(1+a+\ldots+a^{n-2})+a^3(1+a+\ldots+a^{n-3})+$$
$$+a^5(1+a+\ldots+a^{n-4})+\ldots+a^{2n-5}(1+a)+a^{2n-3}=$$
$$=a\frac{a^{n-1}-1}{a-1}+a^3\frac{a^{n-2}-1}{a-1}+a^5\frac{a^{n-3}-1}{a-1}+\ldots+$$
$$+a^{2n-5}\frac{a^2-1}{a-1}+a^{2n-3}\frac{a-1}{a-1}=\frac{1}{a-1}\{(a^n+a^{n+1}+$$
$$+a^{n+2}+\ldots+a^{2n-3}+a^{2n-2})-(a+a^3+a^5+\ldots+a^{2n-5}+$$
$$+a^{2n-3})\}=\frac{(a^n-1)(a^n-a)}{(a-1)(a^2-1)}.$$

39. The sum on the left may be rewritten as follows

$$\left(\frac{1}{x^{n-1}}+\frac{2}{x^{n-2}}+\ldots+\frac{n-1}{x}\right)+[x^{n-1}+2x^{n-2}+\ldots+(n-1)x]+n.$$

The first bracketed expression is equal to

$$\frac{1}{x^n}[x+2x^2+\ldots+(n-1)x^{n-1}]=\frac{x[(n-1)x^n-nx^{n-1}+1]}{x^n(x-1)^2}$$

(see Problem 22).

The second bracketed expression is obtained from the first one by replacing x by $\frac{1}{x}$. Hence, we get the required result.

40. 1° We have

$$\frac{1}{1\cdot 2}=1-\frac{1}{2},$$

$$\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3},$$

$$\frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4}$$

$$\cdots\cdots$$

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Adding the right and left members, we get the required result.

2° The required sum may be rewritten in the following way

$$s=\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}.$$

But $\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\cdot\frac{1}{k}-\frac{1}{k+1}+\frac{1}{2}\cdot\frac{1}{k+2}$.

Therefore

$$s=\frac{1}{2}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+1}+\frac{1}{n+2}\right)-$$
$$-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\frac{1}{n+1}\right)=\frac{1}{2}\left(1+\frac{1}{2}\right)+$$
$$+\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}\right)+\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)-\frac{1}{2}-\frac{1}{n+1}-$$
$$-\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}\right)=\frac{1}{4}+\frac{1}{2}\,\frac{1}{n+2}-\frac{1}{2}\,\frac{1}{n+1}=$$
$$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right).$$

3° Solved as the preceding problem.

41. The sum is equal to

$$S=\sum_{k=1}^{n}\frac{k^4}{4k^2-1}.$$

Hence

$$16S=\sum_{k=1}^{n}\frac{16k^4-1+1}{4k^2-1}=\sum_{k=1}^{n}(4k^2+1)+\frac{1}{2}\sum_{k=1}^{n}\frac{(2k+1)-(2k-1)}{(2k-1)(2k+1)}.$$

$$16S=4\frac{n(n+1)(2n+1)}{6}+n+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right),$$

$$16S=\frac{2n(n+1)(2n+1)}{3}+n+\frac{1}{2}\left\{1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}+\dots+\right.$$
$$\left.+\frac{1}{2n-1}-\frac{1}{2n+1}\right\},$$

$$16S=\frac{2n(n+1)(2n+1)}{3}+n+\frac{n}{2n+1}.$$

Finally

$$16S=\frac{m(m^2+2)}{6}-\frac{1}{2m},$$

where $m=2n+1$.

42. We have

$$\frac{1}{a_1a_n}=\frac{1}{a_1+a_n}\cdot\frac{a_1+a_n}{a_1a_n}=\frac{1}{a_1+a_n}\left(\frac{1}{a_n}+\frac{1}{a_1}\right),$$

$$\frac{1}{a_2a_{n-1}}=\frac{1}{a_2+a_{n-1}}\,\frac{a_2+a_{n-1}}{a_2a_{n-1}}=\frac{1}{a_2+a_{n-1}}\left(\frac{1}{a_2}+\frac{1}{a_{n-1}}\right),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{1}{a_na_1}=\frac{1}{a_1+a_n}\,\frac{a_1+a_n}{a_1a_n}=\frac{1}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_n}\right).$$

But

$$a_1+a_n=a_2+a_{n-1}=a_3+a_{n-2}=\dots.$$

Therefore, adding our equalities termwise, we find

$$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\dots+\frac{1}{a_na_1}=\frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right).$$

43. 1° It is obvious that the following identity takes place

$$\frac{1}{(n+k-1)!}-\frac{1}{(n+k)!}=\frac{n+k-1}{(n+k)!}.$$

Putting $k=1, 2, \dots, p+1$ and adding the obtained equalities termwise, we prove that

$$\frac{n}{(n+1)!}+\frac{n+1}{(n+2)!}+\dots+\frac{n+p}{(n+p+1)!}=\frac{1}{n!}-\frac{1}{(n+p+1)!}.$$

2° We have

$$\frac{n}{(n+1)!}+\frac{n}{(n+2)!}+\ldots+\frac{n}{(n+p+1)!}<\frac{n}{(n+1)!}+$$

$$+\frac{n+1}{(n+2)!}+\ldots+\frac{n+p}{(n+p+1)!}=\frac{1}{n!}-\frac{1}{(n+p+1)!}$$

(see 1°).

Therefore

$$\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots+\frac{1}{(n+p+1)!}<\frac{1}{n}\left\{\frac{1}{n!}-\frac{1}{(n+p+1)!}\right\}$$

44. The following identity holds true

$$\frac{1}{z-1}-\frac{1}{z+1}=\frac{2}{z^2-1}.$$

In our case we have

$$\frac{1}{x-1}-\frac{1}{x+1}=\frac{2}{x^2-1}, \quad (1)$$

$$\frac{1}{x^2-1}-\frac{1}{x^2+1}=\frac{2}{x^4-1}, \quad (2)$$

$$\frac{1}{x^4-1}-\frac{1}{x^4+1}=\frac{2}{x^8-1}, \quad (3)$$

. . . .

$$\frac{1}{x^{2^n}-1}-\frac{1}{x^{2^n}+1}=\frac{2}{x^{2^{n+1}}-1}. \quad (n+1)$$

Multiply both members of equality (1) by 1, of equality (2) by 2, of equality (3) by 2^2 and so forth, finally, multiply both members of the equality $(n+1)$ by 2^n. Adding the obtained results, we find

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\ldots+\frac{2^n}{x^{2^n}+1}=\frac{1}{x-1}-\frac{2^{n+1}}{x^{2^{n+1}}-1}.$$

45. We have

$$\frac{n+p+1}{n-p+1}\sum_{k=1}^{n-p}\frac{n-p-k+1}{(p+k)(n-k+1)}=$$

$$=\frac{1}{n-p+1}\sum_{k=1}^{n-p}\left(\frac{1}{p+k}+\frac{1}{n-k+1}\right)(n-p-k+1)=$$

$$=\frac{1}{n-p+1}\sum_{k=1}^{n-p}\left(\frac{n+1}{p+k}-\frac{p}{n-k+1}\right)=$$

$$=\frac{1}{n-p+1}\left[(n+1)\left(\frac{1}{p+1}+\frac{1}{p+2}+\ldots+\frac{1}{n}\right)-\right.$$

$$\left.-p\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{p+1}\right)\right]=$$

$$=\frac{1}{n-p+1}\left(\frac{1}{p+1}+\ldots+\frac{1}{n}\right)(n+1-p)=$$

$$=\frac{1}{p+1}+\ldots+\frac{1}{n}=\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)-$$

$$-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}\right)=S_n-S_p.$$

46. We have

$$S'_n=\frac{n+1}{2}-\left\{\frac{1}{n(n-1)}+\frac{2}{(n-1)(n-2)}+\ldots+\frac{n-2}{2\cdot 3}\right\}=$$

$$=\frac{n+1}{2}-\sum_{k=1}^{n-2}\frac{k}{(n-k+1)(n-k)}=$$

$$=\frac{n+1}{2}+\sum_{k=1}^{n-2}\frac{-k}{(n-k+1)(n-k)}\,.$$

Let us expand the fraction $\frac{-k}{(n-k+1)(n-k)}$ into two partial fractions. Namely, let us put

$$\frac{-k}{(n-k+1)(n-k)}=\frac{A}{n-k+1}+\frac{B}{n-k}\,,$$

$$-k=A(n-k)+B(n-k+1).$$

Hence, putting first $k=n$ and then $k=n+1$, we find

$$A=n+1,\quad B=-n.$$

Therefore

$$S=\frac{n+1}{2}+(n+1)\sum_{k=1}^{n-2}\frac{1}{n-k+1}-n\sum_{k=1}^{n-2}\frac{1}{n-k}=$$

$$=\frac{n+1}{2}+(n+1)\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{3}\right)-$$

$$-n\left(\frac{1}{n-1}+\frac{1}{n-2}+\ldots+\frac{1}{2}\right)=$$

$$=\frac{n+1}{2}+n\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{3}\right)+$$

$$+\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{3}\right)-$$

$$-n\left[\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{3}\right)-\frac{1}{n}+\frac{1}{2}\right]=$$

$$=\frac{n+1}{2}+\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{3}\right)+1-\frac{n}{2}=$$

$$=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

47. Let the nth term of the required progression be a_n, its common difference being equal to d. Then

$$S_x=\frac{a_1+a_x}{2}\cdot x,$$

$$S_{kx}=\frac{a_1+a_{kx}}{2}kx.$$

Hence

$$\frac{S_{kx}}{S_x}=\frac{a_1+a_{kx}}{a_1+a_x}\cdot k=\frac{2a_1+d(kx-1)}{2a_1+d(x-1)}k=\frac{2a_1-d+kxd}{2a_1-d+dx}\cdot k.$$

For the last relation to have a value independent of x it is necessary and sufficient that

$$2a_1-d=0,$$

i.e. the common difference of the required progression must equal the doubled first term.

48. We can prove the following proposition

$$a_k+a_l=a_{k'}+a_{l'}$$

if $k+l=k'+l'$.

Indeed

$$a_k = a_1 + (k-1)\,d, \qquad a_l = a_1 + (l-1)\,d,$$
$$a_{k'} = a_1 + (k'-1)\,d, \quad a_{l'} = a_1 + (l'-1)\,d.$$

Hence

$$a_k + a_l = 2a_1 + (k+l-2)\,d,$$
$$a_{k'} + a_{l'} = 2a_1 + (k'+l'-2)\,d.$$

But since by hypothesis

$$k + l = k' + l',$$

it follows from the last equalities that

$$a_k + a_l = a_{k'} + a_{l'}.$$

And so we have

$$a_i + a_{i+2} = a_{i+1} + a_{i+1} = 2a_{i+1}.$$

The given sum is therefore transformed as follows

$$S = \sum_{i=1}^{n} \frac{a_i a_{i+1} a_{i+2}}{a_i + a_{i+2}} = \frac{1}{2} \sum_{i=1}^{n} a_i a_{i+2}.$$

But

$$a_i = a_{i+1} - d, \quad a_{i+2} = a_{i+1} + d,$$

therefore

$$S = \frac{1}{2} \sum_{i=1}^{n} (a_{i+1}^2 - d^2) = \frac{1}{2} \sum_{i=1}^{n} [a_1^2 + 2a_1 di + (i^2 - 1)\,d^2] =$$
$$= \frac{1}{2} \left\{ a_1^2 n + 2a_1 d \frac{n(n+1)}{2} - nd^2 + \frac{n(n+1)(2n+1)}{6} d^2 \right\} =$$
$$= \frac{1}{2} n \left\{ a_1^2 + a_1 d\,(n+1) + \frac{(n-1)(2n+5)}{6} d^2 \right\}.$$

49. As is known

$$\tan(\alpha + k\beta) - \tan[\alpha + (k-1)\,\beta] = \frac{\sin\beta}{\cos(\alpha + k\beta)\cos[\alpha + (k-1)\,\beta]}.$$

Therefore

$$\sum_{k=1}^{n} \frac{1}{\cos(\alpha+k\beta)\cos[\alpha+(k-1)\beta]} =$$

$$= \frac{1}{\sin\beta} \sum_{k=1}^{n} \{\tan(\alpha+k\beta) - \tan[\alpha+(k-1)\beta]\} =$$

$$= \frac{1}{\sin\beta} \{\tan(\alpha+\beta) - \tan\alpha + \tan(\alpha+2\beta) - \tan(\alpha+\beta) + \ldots +$$

$$+ \tan(\alpha+n\beta) - \tan(\alpha+(n-1)\beta)\} = \frac{\tan(\alpha+n\beta) - \tan\alpha}{\sin\beta}.$$

50. We have

$$2\cot 2\alpha - \cot\alpha = -\tan\alpha,$$

$$2\cot\alpha - \cot\frac{\alpha}{2} = -\tan\frac{\alpha}{2},$$

$$2\cot\frac{\alpha}{2} - \cot\frac{\alpha}{4} = -\tan\frac{\alpha}{4},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\cot\frac{\alpha}{2^{n-2}} - \cot\frac{\alpha}{2^{n-1}} = -\tan\frac{\alpha}{2^{n-1}}.$$

Multiplying these equalities in turn by $1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^{n-1}}$ and adding termwise, we get the required result.

51. Consider the following formula

$$\cos[a+(k-2)h] - \cos[a+kh] = $$
$$= 2\sin h \sin[a+(k-1)h].$$

Putting $k = 1, 2, 3, \ldots, n-1, n$, we find

$$2\sin h \sin a = \cos(a-h) - \cos(a+h),$$

$$2\sin h \sin(a+h) = \cos a - \cos(a+2h),$$

$$2\sin h \sin(a+2h) = \cos(a+h) - \cos(a+3h),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\sin h \sin[a+(n-2)h] =$$
$$= \cos[a+(n-3)h] - \cos[a+(n-1)h],$$

$$2\sin h \sin[a+(n-1)h] =$$
$$= \cos[a+(n-2)h] - \cos[a+nh].$$

Adding these equalities term by term, we find

$$2\sin h\{\sin a+\sin(a+h)+\sin(a+2h)+\ldots+\sin[a+(n-1)h]\}=\cos a+\cos(a-h)-\cos(a+nh)-\cos[a+(n-1)h]=\{\cos a-\cos[a+(n-1)h]\}+\{\cos(a-h)-\cos(a+nh)\}=$$

$$=2\sin\frac{n-1}{2}h\sin\left(a+\frac{n-1}{2}h\right)+2\sin\left(a+\frac{n-1}{2}h\right)\times\sin\frac{n+1}{2}h=2\sin\left(a+\frac{n-1}{2}h\right)\cdot 2\sin\frac{nh}{2}\cos\frac{h}{2}.$$

Hence

$$\sin a+\sin(a+h)+\sin(a+2h)+\ldots+\sin[a+(n-1)h]=\frac{\sin\left(a+\frac{n-1}{2}h\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

The second formula is obtained similarly. However, it can also be readily obtained from the above deduced formula by replacing a by $\frac{\pi}{2}-a$.

52. Putting in the previous formulas $a=0$, $h=\frac{\pi}{n}$, we get

$$S=\cot\frac{\pi}{2n},\quad S'=0.$$

53. Taking advantage of the results of Problem 51, we have

$$\sin\alpha+\sin 3\alpha+\ldots+\sin[(2n-1)\alpha]=\frac{\sin n\alpha\sin n\alpha}{\sin\alpha},$$

$$\cos\alpha+\cos 3\alpha+\ldots+\cos[(2n-1)\alpha]=\frac{\sin n\alpha\cos n\alpha}{\sin\alpha}.$$

The rest is obvious.

54. The required sums can be computed, for instance, in the following way. Make up the sums S'_n and S''_n. It is easily seen that

$$S'_n+S''_n=2n.$$

On the other hand,

$$S'_n-S''_n=\cos 2x+\cos 4x+\ldots+\cos 4nx.$$

Using the second formula from Problem 51, we find

$$\cos 2x+\cos 4x+\ldots+\cos 4nx=\frac{\sin 2nx\cos(2n+1)x}{\sin x}.$$

And so

$$S'_n-S''_n=\frac{\sin 2nx\cos(2n+1)x}{\sin x}$$

$$S'_n+S''_n=2n.$$

Hence

$$S'_n=n+\frac{\sin 2nx\cos(2n+1)x}{2\sin x},$$

$$S''_n=n-\frac{\sin 2nx\cos(2n+1)x}{2\sin x}.$$

55. Let us make use of the formula

$$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)].$$

We then have

$$S=\sum_{i=1}^{p}\sin\frac{m\pi i}{p+1}\cdot\sin\frac{n\pi i}{p+1}=$$

$$=\frac{1}{2}\sum_{i=1}^{p}\cos\frac{(m-n)\pi i}{p+1}-\frac{1}{2}\sum_{i=1}^{p}\cos\frac{(m+n)\pi i}{p+1}.$$

But if $m+n$ is divisible by $2(p+1)$, then $\cos\frac{(m+n)\pi i}{p+1}=$ $=1$ and

$$S=\frac{1}{2}\sum_{i=1}^{p}\cos\frac{(m-n)\pi i}{p+1}-\frac{1}{2}p.$$

Using formula 2° from Problem 51, we easily find

$$\sum_{i=1}^{p}\cos\frac{(m-n)\pi i}{p+1}=-1.$$

Hence

$$S=-\frac{p+1}{2}.$$

All the remaining cases are proved analogously.

56. We have

$$\arctan (k+1)x+\arctan(-kx)=$$
$$=\arctan\frac{kx+x-kx}{1-(k+1)x(-kx)}=\arctan\frac{x}{1+k(k+1)x^2},$$

since $(k+1)x(-kx)<1$ (see Problem 25, Sec. 3).
Hence

$$\arctan 2x-\arctan x=\arctan\frac{x}{1+1\cdot 2x^2},$$
$$\arctan 3x-\arctan 2x=\arctan\frac{x}{1+2\cdot 3x^2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\arctan (n+1)x-\arctan nx=\arctan\frac{x}{1+n(n+1)x^2}.$$

Adding these equalities termwise, we find that the required sum is equal to

$$\arctan (n+1)x-\arctan x=\arctan\frac{nx}{1+(n+1)x^2}.$$

57. It is obvious that

$$\arctan a_k+\arctan(-a_{k-1})=\arctan\frac{a_k-a_{k-1}}{1+a_ka_{k-1}}=$$
$$=\arctan\frac{r}{1+a_ka_{k-1}}.$$

Now we find easily that our sum is equal to

$$\arctan\frac{a_{n+1}-a_1}{1+a_1a_{n+1}}.$$

58. Put

$$1+k^2+k^4=-xy,\quad x+y=2k.$$

(This is done to use the formula

$$\arctan\frac{x+y}{1-xy}=\arctan x+\arctan y \text{ if } xy<1.)$$

Then

$$\arctan\frac{2k}{2+k^2+k^4}=\arctan(k^2+k+1)-\arctan(k^2-k+1),$$

therefore

$$\sum_{k=1}^{n} \arctan \frac{2k}{2+k^2+k^4} = \arctan 3 - \arctan 1 + \arctan 7 -$$

$$- \arctan 3 + \ldots + \arctan(n^2+n+1) - \arctan(n^2-n+1) =$$

$$= \arctan(n^2+n+1) - \frac{\pi}{4}.$$

59. Let k be one of the numbers $1, 2, \ldots, n-1$. Multiply the first equation by $\sin k\frac{\pi}{n}$, the second by $\sin k\frac{2\pi}{n}$, the third by $\sin k\frac{3\pi}{n}$ and, finally, the last one by $\sin k\frac{(n-1)\pi}{n}$. Adding the obtained products termwise, we find

$$A_1x_1 + A_2x_2 + \ldots + A_{n-1}x_{n-1} = a_1 \sin k\frac{\pi}{n} + a_2 \sin k\frac{2\pi}{n} + \ldots +$$

$$+ a_{n-1} \sin k\frac{(n-1)\pi}{n}.$$

And

$$A_l = \sin l\frac{\pi}{n} \sin k\frac{\pi}{n} + \sin l\frac{2\pi}{n} \sin k\frac{2\pi}{n} + \sin l\frac{3\pi}{n} \sin k\frac{3\pi}{n} +$$

$$+ \ldots + \sin l\frac{(n-1)\pi}{n} \cdot \sin k\frac{(n-1)\pi}{n}.$$

Taking advantage of formula 2° of Problem 51, let us prove that

$$A_l = 0 \quad \text{if} \quad l \neq k,$$

$$A_l = \frac{n}{2} \quad \text{if} \quad l = k.$$

Hence

$$x_k = \frac{2}{n}\left(a_1 \sin k\frac{\pi}{n} + a_2 \cos k\frac{2\pi}{n} + \ldots + a_{n-1} \sin k\frac{(n-1)\pi}{n}\right)$$

$$(k = 1, 2, 3, \ldots, n-1).$$

SOLUTIONS TO SECTION 8

1. We have

$$\frac{1}{2n}=\frac{1}{2n}, \quad \frac{1}{2n-1}>\frac{1}{2n}, \quad \ldots, \quad \frac{1}{n+2}>\frac{1}{2n}, \quad \frac{1}{n+1}>\frac{1}{2n}.$$

Adding these inequalities termwise, we find

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{1}{2n}+\frac{1}{2n}+\ldots+\frac{1}{2n}=\frac{n}{2n}=\frac{1}{2}.$$

2. It is obvious that

$$\frac{1}{(n+k+1)(n+k)}<\frac{1}{(n+k)^2}<\frac{1}{(n+k-1)(n+k)}.$$

But

$$\frac{1}{(n+k+1)(n+k)}=\frac{1}{n+k}-\frac{1}{n+k+1},$$

$$\frac{1}{(n+k-1)(n+k)}=\frac{1}{n+k-1}-\frac{1}{n+k},$$

therefore

$$\frac{1}{n+k}-\frac{1}{n+k+1}<\frac{1}{(n+k)^2}<\frac{1}{n+k-1}-\frac{1}{n+k}.$$

Summing these inequalities (from $k=1$ to $k=p$), we get the required relation.

3. Let us have n fractions ($n\geqslant 1$)

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \ldots, \frac{1}{k}, \frac{1}{l}.$$

Let us assume

$$2\leqslant a<b<c<d<\ldots<k<l.$$

Then

$$b\geqslant a+1, \quad c\geqslant b+1, \quad d\geqslant c+1, \quad \ldots, l\geqslant k+1.$$

Consequently

$$b\geqslant a+1, \quad c\geqslant a+2, \quad d\geqslant a+3, \quad \ldots, \quad l\geqslant a+n-1.$$

Therefore

$$\frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{l^2}\leqslant\frac{1}{a^2}+\frac{1}{(a+1)^2}+\ldots+$$

$$+\frac{1}{(a+n-1)^2}<\frac{1}{a-1}-\frac{1}{a+n-1}.$$

Hence

$$\frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{l^2}<\frac{n}{(a-1)(a+n-1)}.$$

But

$$a-1\geqslant 1,\quad a+n-1\geqslant n+1,\quad (a-1)(a+n-1)\geqslant n+1$$

and

$$\frac{1}{a^2}+\frac{1}{b^2}+\ldots+\frac{1}{l^2}\leqslant\frac{n}{n+1}<1.$$

4. Indeed

$$(n!)^2=(1\cdot n)\cdot(2(n-1))\ldots(n\cdot 1).$$

But

$$k(n-k+1)\geqslant n,$$

since

$$k(n-k+1)-n=(n-k)(k-1)\geqslant 0.$$

Therefore

$$1\cdot n=n,$$
$$2\cdot(n-1)\geqslant n,$$
$$3\cdot(n-2)\geqslant n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$n\cdot 1=n.$$

Hence

$$(n!)^2\geqslant n^n \text{ and } \sqrt[n]{n!}\geqslant\sqrt{n}.$$

5. Since

$$a<\sqrt{A}<a+1,$$

we have

$$\sqrt{A}+a<2a+1,\quad \frac{\sqrt{A}+a}{2a+1}<1,\quad \sqrt{A}-a>0.$$

Hence

$$\frac{(\sqrt{A}+a)(\sqrt{A}-a)}{2a+1}<\sqrt{A}-a,$$

$$\frac{A-a^2}{2a+1}<\sqrt{A}-a,\quad \sqrt{A}>a+\frac{A-a^2}{2a+1}.$$

Let us now prove the second inequality.
For any x there exists the following inequality

$$x(1-x)=x-x^2\leqslant\frac{1}{4}.$$

Indeed, we have

$$x-x^2-\frac{1}{4}=-\left(x-\frac{1}{2}\right)^2\leqslant 0.$$

It is obvious that we have an equality only at $x=\frac{1}{2}$.

Since it is possible to assume that $\sqrt{A}-a\neq\frac{1}{2}$, we have

$$[1-(\sqrt{A}-a)](\sqrt{A}-a)<\frac{1}{4},$$

$$1-(\sqrt{A}-a)<\frac{1}{4(\sqrt{A}-a)},$$

$$(2a+1)-(\sqrt{A}+a)<\frac{1}{4(\sqrt{A}-a)}.$$

Multiplying both members of this inequality by $\sqrt{A}-a>0$, we find

$$(2a+1)(\sqrt{A}-a)-(A-a^2)<\frac{1}{4}.$$

Whence finally

$$\sqrt{A}<a+\frac{A-a^2}{2a+1}+\frac{1}{4(2a+1)}.$$

6. We have

$$\frac{1}{\sqrt{n}}>2\sqrt{n+1}-2\sqrt{n},$$

since

$$\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{2\sqrt{n}}.$$

Consequently,

$$1>2\sqrt{2}-2,$$

$$\frac{1}{\sqrt{2}}>2\sqrt{3}-2\sqrt{2},$$

$$\frac{1}{\sqrt{3}}>2\sqrt{4}-2\sqrt{3},$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{1}{\sqrt{n}}>2\sqrt{n+1}-2\sqrt{n}.$$

Adding these inequalities, we obtain the required result.

7. Put

$$A=\frac{1}{4^s}C_{2s}^{s}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2s-1}{2s}.$$

Then

$$A<\frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2s}{2s+1}=\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2s}{2s-1}\quad\frac{1}{2s+1},$$

i.e.

$$A<\frac{1}{A}\cdot\frac{1}{2s+1}.$$

Hence

$$A^2<\frac{1}{2s+1},\quad A<\frac{1}{\sqrt{2s+1}}.$$

But, on the other hand,

$$A>\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2s-2}{2s-1},$$

$$A=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2s-1}{2s}.$$

Multiplying these relationships, we find

$$A>\frac{1}{2\sqrt{s}}.$$

8. Since

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},$$

we have

$$\cot\theta=\frac{1-\frac{1}{\cot^2\frac{\theta}{2}}}{2\frac{1}{\cot\frac{\theta}{2}}}=\frac{\cot^2\frac{\theta}{2}-1}{2\cot\frac{\theta}{2}}.$$

Consequently

$$1+\cot\theta-\cot\frac{\theta}{2}=1+\frac{\cot^2\frac{\theta}{2}-1}{2\cot\frac{\theta}{2}}-\cot\frac{\theta}{2}=$$

$$=\frac{-1}{2\cot\frac{\theta}{2}}\left\{\cot^2\frac{\theta}{2}-2\cot\frac{\theta}{2}+1\right\}=-\frac{\left(1-\cot\frac{\theta}{2}\right)^2}{2\cot\frac{\theta}{2}}\leqslant 0,$$

since

$$\cot \frac{\theta}{2} > 0 \; (0 < \theta < \pi).$$

9. We have

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(\pi - C) = -\tan C > 0,$$

since C is an obtuse angle.

And so

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} > 0.$$

But since A and B are less than $\frac{\pi}{2}$, it follows that $\tan A + \tan B > 0$, and hence

$$1 - \tan A \tan B > 0, \quad \tan A \tan B < 1.$$

10. Indeed

$$\tan(\theta - \varphi) = \frac{\tan\theta - \tan\varphi}{1 + \tan\theta \tan\varphi} = \frac{(n-1)\tan\varphi}{1 + n\tan^2\varphi}.$$

Therefore

$$\tan^2(\theta - \varphi) = \frac{(n-1)^2}{(\cot\varphi + n\tan\varphi)^2} = \frac{(n-1)^2}{(\cot\varphi - n\tan\varphi)^2 + 4n} \leqslant \frac{(n-1)^2}{4n}.$$

11. We have

$$\cos 2\gamma = \frac{1 - \tan^2\gamma}{1 + \tan^2\gamma}.$$

To prove that $\cos 2\gamma \leqslant 0$, it is sufficient to prove that

$$1 - \tan^2\gamma \leqslant 0.$$

But we have

$$1 - \tan^2\gamma = \frac{\cos^2\alpha\cos^2\beta - (1 + \sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}.$$

We only have to prove that

$$\cos^2\alpha\cos^2\beta - (1 + \sin\alpha\sin\beta)^2 \leqslant 0.$$

But

$$\cos^2\alpha\cos^2\beta - (1 + \sin\alpha\sin\beta)^2 =$$
$$= (1 - \sin^2\alpha)(1 - \sin^2\beta) - (1 + \sin\alpha\sin\beta)^2 =$$
$$= -(\sin\alpha + \sin\beta)^2 \leqslant 0.$$

12. Let m be the least and M the greatest of the given fractions.

Then

$$m \leqslant \frac{a_i}{b_i} \leqslant M \quad (i = 1, 2, 3, \ldots, n).$$

Hence

$$mb_i \leqslant a_i \leqslant Mb_i.$$

Summing all these inequalities (from $i = 1$ to $i = n$), we find

$$m \sum b_i \leqslant \sum a_i \leqslant M \sum b_i.$$

And so indeed

$$m \leqslant \frac{\sum a_i}{\sum b_i} \leqslant M.$$

13. We assume, of course, that all the quantities $a, b, \ldots, l$ are positive, and the principal value of the root is taken everywhere. Besides, $m, n, \ldots, p$ are positive integers. Let us take logarithms of our roots, i.e. consider the quantities

$$\frac{\log a}{m}, \quad \frac{\log b}{n}, \quad \ldots, \frac{\log l}{p}.$$

Let μ be the least and M the greatest of these fractions. On the basis of the results of Problem 12 we have

$$\mu < \frac{\log a + \log b + \ldots + \log l}{m + n + \ldots + p} < M.$$

Consequently

$$\mu < \log \sqrt[m+n+\ldots+p]{ab \ldots l} < M,$$

wherefrom follows our proposition.

14. See Problem 12.

15. We have

$$x^{\lambda} - y^{\lambda} - z^{\lambda} = y^2 (x^{\lambda-2} - y^{\lambda-2}) + z^2 (x^{\lambda-2} - z^{\lambda-2}),$$

since

$$x^2 = y^2 + z^2.$$

From the same equality follow $x > y$, $x > z$. Therefore, if

$$\lambda - 2 > 0,$$

then

$$x^{\lambda-2} - y^{\lambda-2} > 0 \quad \text{and} \quad x^{\lambda-2} - z^{\lambda-2} > 0,$$

and, consequently, for $\lambda > 2$,

$$x^{\lambda} - y^{\lambda} - z^{\lambda} > 0, \quad \text{i.e.} \quad x^{\lambda} > y^{\lambda} + z^{\lambda}.$$

We prove in the same way that

$$x^{\lambda} < y^{\lambda} + z^{\lambda} \quad \text{if} \quad \lambda < 2.$$

16. (See Problem 7, Sec. 1). It can be proved, for instance, in the following manner. If $a^2 + b^2 = 1$, then, obviously, we can find an angle φ such that

$$a = \cos\varphi, \quad b = \sin\varphi.$$

Likewise we can find an angle φ' such that

$$m = \cos\varphi', \quad n = \sin\varphi'.$$

Then we have

$$|am + bn| = |\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'| = |\cos(\varphi - \varphi')| \leqslant 1.$$

17. We have

$$a^2 \geqslant a^2 - (b - c)^2,$$
$$b^2 \geqslant b^2 - (c - a)^2,$$
$$c^2 \geqslant c^2 - (a - b)^2.$$

Multiplying, we get

$$a^2b^2c^2 \geqslant (a + b - c)^2 (a + c - b)^2 (b + c - a)^2.$$

Hence follows the required inequality.

18. It is known that if $A + B + C = \pi$, then

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1$$

(see Problem 40, 4°, Sec. 2).

Put

$$\tan\frac{A}{2} = x, \quad \tan\frac{B}{2} = y, \quad \tan\frac{C}{2} = z.$$

It only remains to prove that

$$x^2 + y^2 + z^2 \geqslant 1$$

if

$$xy + xz + yz = 1.$$

But we have

$$2(x^2 + y^2 + z^2) - 2(xy + xz + yz) =$$
$$= (x - y)^2 + (x - z)^2 + (y - z)^2 \geqslant 0.$$

Hence

$$2(x^2 + y^2 + z^2) - 2 \leqslant 0,$$
$$x^2 + y^2 + z^2 \geqslant 1.$$

19. We have

$$\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}, \quad \sin\frac{B}{2} = \sqrt{\frac{(p-a)(p-c)}{ac}},$$
$$\sin\frac{C}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}}.$$

Consequently, it is sufficient to prove that

$$\frac{(p-a)(p-b)(p-c)}{abc} \leqslant \frac{1}{8}.$$

But

$$p - a = \frac{a+b+c}{2} - a = \frac{b+c-a}{2}, \quad p - b = \frac{a+c-b}{2},$$
$$p - c = \frac{a+b-c}{2}.$$

Therefore, we have to prove only the following

$$\frac{(b+c-a)(a+c-b)(a+b-c)}{abc} \leqslant 1,$$

provided $b+c-a > 0$, $a+c-b > 0$ and $a+b-c > 0$ (see Problem 17). This inequality can be proved in a different way. Put

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \xi;$$

then we have

$$\xi = \frac{1}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\cos\frac{A+B}{2}.$$

Hence

$$\cos^2\frac{A+B}{2} - \cos\frac{A-B}{2}\cos\frac{A+B}{2} + 2\xi = 0.$$

Consequently

$$\cos\frac{A+B}{2}=\frac{\cos\frac{A-B}{2}\pm\sqrt{\cos^2\frac{A-B}{2}-8\xi}}{2}.$$

Since $\cos\frac{A+B}{2}$ and $\cos\frac{A-B}{2}$ are real, there must be

$$\cos^2\frac{A-B}{2}-8\xi\geqslant 0,$$

$$8\xi\leqslant\cos^2\frac{A-B}{2},\quad 8\xi\leqslant 1,\ \xi\leqslant\frac{1}{8}.$$

20. 1° We have the relationship (see Problem 40, 2°, Sec. 2)

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Using the result of the preceding problem, we get the required inequality.

2° Since there exists the following relationship

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{1}{4}(\sin A+\sin B+\sin C),$$

the given problem represents a particular case of Problem 48 of this section.

21. It is sufficient to prove that

$$(a+c)(b+d)\geqslant ab+cd+2\sqrt{abcd},$$

i.e. that

$$cb+ad\geqslant 2\sqrt{cbad}.$$

But

$$cb+ad-2\sqrt{cbad}=(\sqrt{cb}-\sqrt{ad})^2\geqslant 0.$$

22. We have

$$a^2+b^2-2ab=(a-b)^2\geqslant 0.$$

Hence

$$a^2-ab+b^2\geqslant ab,$$

$$a^3+b^3\geqslant ab(a+b).$$

Consequently

$$3a^3+3b^3\geqslant 3a^2b+3ab^2.$$

Add a^3+b^3 to both members of the last inequality.

We have

$$4a^3+4b^3 \geqslant (a+b)^3.$$

And so, indeed,

$$\frac{a^3+b^3}{2} \geqslant \left(\frac{a+b}{2}\right)^3.$$

23. 1° It is required to prove that the arithmetic mean of two positive numbers is not less than their geometric mean. Indeed,

$$\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}(a+b-2\sqrt{ab})=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2 \geqslant 0.$$

2° To prove that

$$\frac{a+b}{2}-\sqrt{ab} \leqslant \frac{1}{8}\frac{(a-b)^2}{b} \quad (a>b),$$

it is sufficient to prove that

$$\frac{(\sqrt{a}-\sqrt{b})^2}{2} \leqslant \frac{1}{8}\frac{(a-b)^2}{b}.$$

Consequently, it is necessary to prove the following

$$\frac{(\sqrt{a}+\sqrt{b})^2}{8b} \geqslant \frac{1}{2}.$$

We have

$$\frac{(\sqrt{a}+\sqrt{b})^2}{8b}=\frac{1}{8}\left(1+\sqrt{\frac{a}{b}}\right)^2 \geqslant \frac{1}{2},$$

since $\frac{a}{b}>1$.

The second inequality is proved in a similar way.

24. Put $a=x^3$, $b=y^3$, $c=z^3$. The only thing to be proved is that

$$x^3+y^3+z^3-3xyz \geqslant 0$$

for any non-negative x, y and z.

But we have (see Problem 20, Sec. 1)

$$x^3+y^3+z^3-3xyz=(x+y+z)\times$$
$$\times(x^2+y^2+z^2-xy-xz-yz).$$

And so, it only remains to prove that

$$x^2+y^2+z^2-xy-xz-yz \geqslant 0.$$

But we have (see Problem 10, Sec. 5)

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz =$$
$$= (x - y)^2 + (x - z)^2 + (y - z)^2 \geqslant 0.$$

25. We have

$$\sqrt{a_1a_2} \leqslant \frac{a_1 + a_2}{2}, \quad \sqrt{a_1a_3} \leqslant \frac{a_1 + a_3}{2}, \quad \ldots, \quad \sqrt{a_{n-1}a_n} \leqslant \frac{a_{n-1} + a_n}{2}.$$

Adding them termwise, we get the required inequality.

26. We have

$$\frac{1 + a_1}{2} \geqslant \sqrt{a_1}, \quad \frac{1 + a_2}{2} \geqslant \sqrt{a_2}, \quad \ldots, \quad \frac{1 + a_n}{2} \geqslant \sqrt{a_n}.$$

Multiplying these inequalities term by term, we have

$$\frac{(1 + a_1)(1 + a_2) \ldots (1 + a_n)}{2^n} \geqslant \sqrt{a_1a_2 \ldots a_n} = 1.$$

And so, indeed,

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geqslant 2^n.$$

27. 1° Make use of the following identity

$$(a + b)(a + c)(b + c) =$$
$$= (ab + ac + bc)(a + b + c) - abc.$$

But

$$\frac{a + b + c}{3} \geqslant \sqrt[3]{abc}, \quad \frac{ab + ac + bc}{3} \geqslant \sqrt[3]{a^2b^2c^2}.$$

Therefore

$$(a + b + c)(ab + ac + bc) \geqslant 9abc,$$

and consequently

$$(a + b)(a + c)(b + c) \geqslant 8abc.$$

2° We have

$$\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} = \frac{a + b + c}{b + c} - 1 + \frac{b + a + c}{a + c} - 1 +$$
$$+ \frac{c + a + b}{a + b} - 1 = (a + b + c)\left(\frac{1}{b + c} + \frac{1}{a + c} + \frac{1}{a + b}\right) - 3$$

But

$$(b + c) + (a + c) + (c + b) \geqslant 3\sqrt[3]{(b + c)(a + c)(a + b)},$$

i.e.

$$a + b + c \geqslant \frac{3}{2}\sqrt[3]{(b + c)(a + c)(a + b)}.$$

Further

$$\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}=\frac{1}{(b+c)(a+c)(a+b)}\{(b+c)(a+c)+$$
$$+(b+c)(a+b)+(a+b)(a+c)\}\geqslant$$
$$\geqslant\frac{3}{(b+c)(a+c)(a+b)}\sqrt[3]{(b+c)^2(a+c)^2(a+b)^2}.$$

Therefore

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geqslant\frac{3}{2}\sqrt[3]{(b+c)(a+c)(a+b)}\times$$
$$\times\frac{3}{(b+c)(a+c)(a+b)}\sqrt[3]{(b+c)^2(a+c)^2(a+b)^2}-3.$$

Thus

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geqslant\frac{3}{2}.$$

28. It is sufficient to prove that

$$(a+k)(b+l)(c+m)\geqslant(\sqrt[3]{abc}+\sqrt[3]{klm})^3.$$

We have

$$(a+k)(b+l)(c+m)=$$
$$=abc+klm+(alc+kbc+abm)+(klc+alm+kbm),$$
$$(\sqrt[3]{abc}+\sqrt[3]{klm})^3=abc+klm+$$
$$+3\sqrt[3]{a^3b^2c^2klm}+3\sqrt[3]{k^2l^2m^2abc}.$$

But

$$\frac{alc+kbc+abm}{3}\geqslant\sqrt[3]{a^2b^2c^2klm},\qquad\frac{klc+alm+kbm}{3}\geqslant\sqrt[3]{k^2l^2m^2abc}.$$

Hence follows the validity of our inequality.

29. We have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geqslant3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}=\frac{3}{\sqrt[3]{abc}}.$$

But

$$\sqrt[3]{abc}\leqslant\frac{a+b+c}{3},$$

i.e.

$$\frac{1}{\sqrt[3]{abc}}\geqslant\frac{3}{a+b+c}.$$

Therefore

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geqslant 3\frac{1}{\sqrt[3]{abc}}\geqslant\frac{9}{a+b+c}.$$

30. It is necessary to prove that the arithmetic mean of n positive numbers is not less ($\geqslant$) than the geometric mean of these numbers. We are going through several proofs of this proposition. Let us begin with the most elegant one which belongs to Cauchy.

Thus, we have to prove that

$$\frac{x_1+x_2+\ldots+x_n}{n}\geqslant\sqrt[n]{x_1x_2\ldots x_n}.$$

At $n=1$ the validity of this inequality is obvious. At $n=2$ and $n=3$ the proposition was proved in Problems 23 and 24.

Let us first show how to prove the validity of our assertion at $n=4$. We have

$$\frac{x_1+x_2+x_3+x_4}{4}=\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}}{2}\geqslant\sqrt{\frac{x_1+x_2}{2}\cdot\frac{x_3+x_4}{2}}.$$

But

$$\frac{x_1+x_2}{2}\geqslant\sqrt{x_1x_2},\qquad\frac{x_3+x_4}{2}\geqslant\sqrt{x_3x_4}.$$

Therefore

$$\frac{x_1+x_2+x_3+x_4}{4}\geqslant\sqrt{\sqrt{x_1x_2}\cdot\sqrt{x_3x_4}}=\sqrt[4]{x_1x_2x_3x_4}.$$

Let us now prove that, in general, if the theorem holds at $n=m$, then it is valid at $n=2m$ too.

Indeed,

$$\frac{x_1+x_2+x_3+\ldots+x_{2m-1}+x_{2m}}{2m}=$$

$$=\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}+\ldots+\frac{x_{2m-1}+x_{2m}}{2}}{m}\geqslant$$

$$\geqslant\sqrt[m]{\frac{x_1+x_2}{2}\cdot\frac{x_3+x_4}{2}\ldots\frac{x_{2m-1}+x_{2m}}{2}}$$

(since we assume that the theorem is valid at $n=m$).

Further

$$\frac{x_1+x_2+x_3+\ldots+x_{2m}}{2m} \geqslant$$

$$\geqslant \sqrt[m]{\sqrt{x_1x_2}\cdot\sqrt{x_3x_4}\ldots\sqrt{x_{2m-1}x_{2m}}} = \sqrt[2m]{x_1x_2x_3x_4\ldots x_{2m}}.$$

And so, assuming that the theorem is valid at $n = m$, we have proved that it is true at $n = 2m$ as well. And since we proved the validity of the theorem for $n = 2$, it is valid for $n = 4, 8, 16, \ldots$, i.e. for n equal to any power of two. However, we have to prove that the theorem is true for any whole n. Let us take some value of n. If n is a power of two, then for such a value of n the theorem is valid, if not, then it is always possible to add a certain q to n such that $n + q$ will yield some power of two.

Put

$$n+q=2^m.$$

We then have

$$\frac{x_1+x_2+x_3+\ldots+x_n+x_{n+1}+\ldots+x_{n+q}}{n+q} \geqslant$$

$$\geqslant \sqrt[n+q]{x_1x_2\ldots x_nx_{n+1}\ldots x_{n+q}}$$

for any positive x_i $(i=1, 2, \ldots, n+q)$.

Put

$$x_{n+1}=x_{n+2}=\ldots=x_{n+q}=\frac{x_1+x_2+\ldots+x_n}{n}.$$

We get

$$\frac{x_1+x_2+\ldots+x_n+\frac{x_1+x_2+\ldots+x_n}{n}\cdot q}{n+q} \geqslant$$

$$\geqslant \sqrt[n+q]{x_1x_2\ldots x_n\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^q}.$$

Hence

$$\frac{x_1+x_2+\ldots+x_n}{n} \geqslant \sqrt[n+q]{x_1x_2\ldots x_n\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^q}$$

or

$$\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^{n+q} \geqslant x_1x_2\ldots x_n\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^q,$$

$$\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^n \geqslant x_1x_2\ldots x_n$$

and finally

$$\frac{x_1+x_2+\ldots+x_n}{n} \geqslant \sqrt[n]{x_1x_2\ldots x_n}.$$

And so, the theorem is valid for any whole n. It is obvious that if $x_1 = x_2 = \ldots = x_n$, then the sign of equality takes place in our theorem. Let us prove that *the sign of equality occurs only* when all the quantities $x_1, x_2, \ldots, x_n$ are equal to one another. Suppose at least two of them, say x_1 and x_2, are not equal to each other. Let us prove that in this case only the sign of inequality is possible, i.e. it will be

$$\frac{x_1+x_2+\ldots+x_n}{n} > \sqrt[n]{x_1x_2\ldots x_n}.$$

Indeed

$$\frac{x_1+x_2+\ldots+x_n}{n} = \frac{\frac{x_1+x_2}{2}+\frac{x_1+x_2}{2}+x_3+\ldots+x_n}{n} \geqslant$$

$$\geqslant \sqrt[n]{\left(\frac{x_1+x_2}{2}\right)^2 x_3\ldots x_n}.$$

But if x_1 is not equal to x_2, then

$$\frac{x_1+x_2}{2} > \sqrt{x_1x_2},$$

consequently

$$\sqrt[n]{\left(\frac{x_1+x_2}{2}\right)^2 x_3\ldots x_n} > \sqrt[n]{x_1x_2x_3\ldots x_n},$$

and therefore

$$\frac{x_1+x_2+\ldots+x_n}{n} > \sqrt[n]{x_1x_2\ldots x_n}$$

if at least two of the quantities $x_1, x_2, \ldots, x_n$ are not equal to one another.

Given below are some more proofs of this theorem. Let us pass over to the *second* one. Let n be a positive number greater than or equal to unity ($n \geqslant 1$). We assume here that a and b are two real positive numbers. Then the following inequality takes place

$$(a^{n-1}-b^{n-1})(a-b) \geqslant 0,$$

Hence

$$a^n+b^n \geqslant a^{n-1}b+b^{n-1}a.$$

Consider n positive numbers $a, b, c, \ldots, k, l$. Let us apply this inequality to all possible pairs of numbers made up of the given n numbers. Adding the inequalities thus obtained, we find

$$\begin{aligned}(a^n+b^n)+(a^n+c^n)+\ldots+(a^n+l^n)+\\ +(b^n+c^n)+\ldots+(b^n+l^n)+\ldots+(k^n+l^n)\geqslant\\ \geqslant(a^{n-1}b+b^{n-1}a)+(a^{n-1}c+c^{n-1}a)+\ldots+\\ +(a^{n-1}l+l^{n-1}a)+\ldots+(k^{n-1}l+l^{n-1}k).\end{aligned}$$

Hence we have

$$\begin{aligned}(n-1)(a^n+b^n+\ldots+l^n)\geqslant\\ \geqslant a(b^{n-1}+c^{n-1}+\ldots+l^{n-1})+b(a^{n-1}+c^{n-1}+\ldots+l^{n-1})+\\ +c(a^{n-1}+b^{n-1}+\ldots+l^{n-1})+\ldots+\\ +l(a^{n-1}+b^{n-1}+\ldots+k^{n-1}). \quad (*)\end{aligned}$$

Using this inequality, it is possible to prove our theorem on the relation between the arithmetic and geometric means of n numbers by the method of induction. We have to prove that

$$\frac{x_1+x_2+\ldots+x_n}{n}\geqslant\sqrt[n]{x_1x_2\ldots x_n}.$$

Put

$$x_1=a^n, \quad x_2=b^n, \quad x_3=c^n, \quad \ldots, \quad x_{n-1}=k^n, \quad x_n=l^n.$$

Then it is sufficient to prove that

$$\frac{a^n+b^n+\ldots+k^n+l^n}{n}\geqslant ab\ldots kl.$$

Let us assume that this inequality is valid at the exponent equal to $n-1$, i.e.

$$\begin{aligned}b^{n-1}+\ldots+k^{n-1}+l^{n-1}\geqslant(n-1)\,b\cdot k\ldots l,\\ a^{n-1}+c^{n-1}+\ldots+l^{n-1}\geqslant(n-1)\,a\cdot c\ldots l,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a^{n-1}+b^{n-1}+\ldots+k^{n-1}\geqslant(n-1)\,a\cdot b\ldots k\end{aligned}$$

Using the inequality (*), we find

$$(n-1)(a^n+b^n+\ldots+k^n+l^n)\geqslant$$
$$\geqslant a(n-1)bk\ldots l+b(n-1)ac\ldots l+\ldots+$$
$$+l(n-1)ab\ldots k.$$

Hence

$$(n-1)(a^n+b^n+\ldots+k^n+l^n)\geqslant(n-1)\cdot n\cdot abc\ldots kl,$$

i.e.

$$\frac{a^n+b^n+\ldots+l^n}{n}\geqslant abc\ldots kl.$$

Thus, our theorem is proved for the second time. Let us pass over to the *third* proof of this theorem. It will be carried out using the method of mathematical induction once again. Let there be n positive numbers $a, b, \ldots, k, l$. It is required to prove that

$$a+b+\ldots+k+l\geqslant n\sqrt[n]{ab\ldots kl}.$$

Assuming that the theorem holds true for $n-1$ numbers, we have

$$a+b+\ldots+k+l\geqslant(n-1)\sqrt[n-1]{ab\ldots k}+l.$$

And so, the theorem will be proved if we prove the inequality

$$(n-1)\sqrt[n-1]{ab\ldots k}+l\geqslant n\sqrt[n]{ab\ldots k\cdot l}.$$

Thus, we have to prove the inequality

$$(n-1)\sqrt[n-1]{\frac{ab\ldots kl}{l^n}}+1\geqslant n\sqrt[n]{\frac{ab\ldots kl}{l^n}}.$$

Put

$$\frac{ab\ldots kl}{l^n}=\xi^{n(n-1)}.$$

Therefore, it is required to prove that

$$(n-1)\xi^n+1\geqslant n\xi^{n-1}.$$

And so, to prove our theorem means to prove the inequality

$$n\xi^{n-1}(\xi-1)\geqslant\xi^n-1,$$

where ξ is any real positive number and n is a positive integer. Let us prove this inequality. At $\xi = 1$ we obviously have the equality. Suppose now $\xi > 1$. It is required to prove that

$$\frac{\xi^n - 1}{\xi - 1} \leqslant n\xi^{n-1}.$$

We have

$$\frac{\xi^n - 1}{\xi - 1} = \xi^{n-1} + \xi^{n-2} + \ldots + \xi^2 + \xi + 1.$$

But

$$1 < \xi < \xi^2 < \xi^3 < \ldots < \xi^{n-2} < \xi^{n-1}.$$

Therefore

$$\xi^{n-1} + \xi^{n-2} + \ldots + \xi + 1 < n\xi^{n-1},$$

and, consequently, indeed

$$\frac{\xi^n - 1}{\xi - 1} < n\xi^{n-1}.$$

If $\xi < 1$, we have to prove that

$$\frac{\xi^n - 1}{\xi - 1} > n\xi^{n-1}.$$

This result is obtained as in the previous case, and, thus, the theorem is proved.

All the considered proofs were carried out using the method of mathematical induction. Therefore, it is desirable to get such a proof which would establish immediately that

$$\frac{a_1 + a_2 + \ldots + a_n}{n} > \sqrt[n]{a_1 a_2 \ldots a_n}$$

if $a_1, a_2, \ldots, a_n$ are any positive quantities not equal to one another simultaneously. Put $a_i = x_i^n$. Then we have to prove that

$$\frac{x_1^n + x_2^n + \ldots + x_n^n}{n} - x_1 x_2 \ldots x_n > 0,$$

i.e. the problem is reduced to finding out that a certain function (form) of n variables $x_1, x_2, \ldots, x_n$ is positive. As is known, n letters $x_1, x_2, \ldots, x_n$ can be permutated by $n!$ methods. If $f(x_1, x_2, \ldots, x_n)$ is a function of n variables $x_1, x_2, \ldots, x_n$, then the symbol $\sum f(x_1, x_2, \ldots, x_n)$ will denote the sum of $n!$ quantities obtained from $f(x_1, x_2, \ldots, x_n)$, using, all possible permutations. For example,

$$\sum x_1x_2 \ldots x_n = n!\, x_1x_2 \ldots x_n,$$

$$\sum x_1^n = (n-1)!\,(x_1^n + x_2^n + \ldots + x_n^n).$$

Introduce the notation

$$\frac{x_1^n + x_2^n + \ldots + x_n^n}{n} - x_1x_2 \ldots x_n = \varphi(x_1, x_2, \ldots, x_n).$$

It is easily seen, that whatever permutation is used, the function $\varphi(x_1, x_2, \ldots, x_n)$ remains unchanged. Therefore we have

$$n!\,\varphi(x_1, x_2, \ldots, x_n) = \\ = \frac{1}{n} \sum (x_1^n + x_2^n + \ldots + x_n^n) - \sum x_1x_2 \ldots x_n$$

But

$$\sum x_1^n + x_2^n + \ldots + x_n^n = n!\,(x_1^n + x_2^n + \ldots + x_n^n).$$

On the other hand,

$$x_1^n + x_2^n + \ldots + x_n^n = \frac{1}{(n-1)!} \sum x_1^n,$$

therefore

$$n!\,\varphi(x_1, x_2, \ldots, x_n) = \sum x_1^n - \sum x_1x_2 \ldots x_n. \qquad (*)$$

Let us consider the following functions

$$\varphi_1 = \sum (x_1^{n-1} - x_2^{n-1})(x_1 - x_2),$$

$$\varphi_2 = \sum (x_1^{n-2} - x_2^{n-2})(x_1 - x_2)\,x_3,$$

$$\varphi_3 = \sum (x_1^{n-3} - x_2^{n-3})(x_1 - x_2)\,x_3x_4,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\varphi_{n-1} = \sum (x_1 - x_2)(x_1 - x_2)\,x_3x_4 \ldots x_n.$$

We have

$$\varphi_1 = 2\sum x_1^n - 2\sum x_1^{n-1}x_2,$$
$$\varphi_2 = 2\sum x_1^{n-1}x_2 - 2\sum x_1^{n-2}x_2x_3,$$
$$\varphi_3 = 2\sum x_1^{n-2}x_2x_3 - 2\sum x_1^{n-3}x_2x_3x_4,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varphi_{n-1} = 2\sum x_1^2x_2x_3\ldots x_n - 2\sum x_1x_2x_3\ldots x_n.$$

Adding these expressions termwise, we find

$$\varphi_1+\varphi_2+\varphi_3+\ldots+\varphi_{n-1} = 2\sum x_1^n - 2\sum x_1x_2\ldots x_n.$$

Comparing this with the equality (*), we get

$$n!\,\varphi(x_1, x_2, \ldots, x_n) = \frac{1}{2}(\varphi_1+\varphi_2+\varphi_3+\ldots+\varphi_{n-1}).$$

And so

$$\frac{x_1^n+x_2^n+\ldots+x_n^n}{n} - x_1x_2\ldots x_n = \frac{1}{2\cdot n!}(\varphi_1+\varphi_2+\ldots+\varphi_{n-1}).$$

But it is evident that $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ vanish if and only if $x_1 = x_2 = \ldots = x_n$.

If not all of the variables are simultaneously equal to one another, then all $\varphi_i > 0$. Indeed, we have

$$\varphi_1 = \sum (x_1-x_2)^2(x_1^{n-2}+\ldots+x_2^{n-2}) \geqslant 0,$$
$$\varphi_2 = \sum (x_1-x_2)^2(x_1^{n-3}+\ldots+x_2^{n-3})\,x_3 \geqslant 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varphi_{n-1} = \sum (x_1-x_2)^2 x_3x_4\ldots x_n \geqslant 0.$$

Therefore

$$\frac{x_1^n+x_2^n+\ldots+x_n^n}{n} - x_1x_2\ldots x_n \geqslant 0,$$

the equality being possible only if $x_1 = x_2 = \ldots = x_n$. And so, the theorem is proved. This proof belongs to A. Gurwitz.

31. We have (using the preceding problem)

$$\sqrt[n]{a_1a_2\ldots a_n} \leqslant \frac{a_1+a_2+\ldots+a_n}{n} = \frac{a_1+a_n}{2n}\,n = \frac{a_1+a_n}{2}.$$

To prove the second inequality consider the product

$$(a_1a_2\ldots a_n)^2 = (a_1a_n)(a_2a_{n-1})\ldots(a_na_1).$$

But we can prove that

$$a_k a_{n-k+1} \geqslant a_1 a_n \quad \text{(see Problem 19, Sec. 7)}$$

Therefore

$$(a_1 a_2 \ldots a_n)^2 \geqslant (a_1 a_n)^n$$

and

$$\sqrt[n]{a_1 a_2 \ldots a_n} \geqslant \sqrt{a_1 a_n}.$$

32. Consider a quantities equal to $\frac{1}{a}$, b quantities equal to $\frac{1}{b}$, and c quantities equal to $\frac{1}{c}$. The arithmetic mean of these quantities will be

$$\frac{a\cdot\frac{1}{a}+b\cdot\frac{1}{b}+c\cdot\frac{1}{c}}{a+b+c}=\frac{3}{a+b+c}.$$

The geometric mean is equal to

$$\sqrt[a+b+c]{\frac{1}{a^a}\cdot\frac{1}{b^b}\cdot\frac{1}{c^c}}.$$

Consequently

$$\frac{3}{a+b+c} \geqslant \sqrt[a+b+c]{\frac{1}{a^a}\cdot\frac{1}{b^b}\cdot\frac{1}{c^c}},$$

i.e.

$$\frac{a}{a^{a+b+c}}\,\frac{b}{b^{a+b+c}}\,\frac{c}{c^{a+b+c}} \geqslant \frac{1}{3}(a+b+c).$$

33. Put

$$a=\frac{\alpha}{m}, \quad b=\frac{\beta}{m}, \quad c=\frac{\gamma}{m},$$

where α, β, γ and m are positive integers.

Consider the product

$$\left(1+\frac{b-c}{a}\right)^a\left(1+\frac{c-a}{b}\right)^b\left(1+\frac{a-b}{c}\right)^c=$$

$$=\sqrt[m]{\left(1+\frac{b-c}{a}\right)^\alpha\left(1+\frac{c-a}{b}\right)^\beta\left(1+\frac{a-b}{c}\right)^\gamma}.$$

Since α, β and γ are whole positive integers, the radicand may be considered as a product of α factors equal to $1+\frac{b-c}{a}$ each, β factors equal to $1+\frac{c-a}{b}$ each, and γ

factors equal to $1+\frac{a-b}{c}$ each. Then we have

$$\sqrt[\alpha+\beta+\gamma]{\left(1+\frac{b-c}{a}\right)^{\alpha}\left(1+\frac{c-a}{b}\right)^{\beta}\left(1+\frac{a-b}{c}\right)^{\gamma}}\leqslant$$

$$\leqslant\frac{\alpha\left(1+\frac{b-c}{a}\right)+\beta\left(1+\frac{c-a}{b}\right)+\gamma\left(1+\frac{a-b}{c}\right)}{\alpha+\beta+\gamma}=1.$$

Raising both members of this inequality to the power $a+b+c$, we get the required result.

34. We have

$$\frac{\frac{s}{s-a}+\frac{s}{s-b}+\ldots+\frac{s}{s-l}}{n}\geqslant$$

$$\geqslant\sqrt[n]{\frac{s^n}{(s-a)(s-b)\ldots(s-l)}}=\frac{s}{\sqrt[n]{(s-a)(s-b)\ldots(s-l)}}.$$

But

$$\sqrt[n]{(s-a)(s-b)\ldots(s-l)}\leqslant$$

$$\leqslant\frac{(s-a)+(s-b)+\ldots+(s-l)}{n}=\frac{n-1}{n}\cdot s.$$

Therefore

$$\frac{1}{\sqrt[n]{(s-a)(s-b)\ldots(s-l)}}\geqslant\frac{n}{(n-1)s}.$$

The further proof is obvious.

35. First of all this inequality can be obtained from Lagrange's identity (see Problem 5, Sec. 1). But we shall proceed in a somewhat different way. Let us set up the following expression

$$(\lambda a_1+\mu b_1)^2+(\lambda a_2+\mu b_2)^2+\ldots+(\lambda a_n+\mu b_n)^2=$$

$$=A\lambda^2+2B\lambda\mu+C\mu^2,$$

where

$$A=a_1^2+a_2^2+\ldots+a_n^2,\quad C=b_1^2+b_2^2+\ldots+b_n^2,$$

$$B=a_1b_1+a_2b_2+\ldots+a_nb_n.$$

Since the left member of this inequality represents the sum of squares, we have

$$A\lambda^2+2B\lambda\mu+C\mu^2\geqslant 0.$$

for all values of λ and μ.

Consequently, the trinomial

$$Ax^2 + 2Bx + C$$

is greater than or equal to zero for all real values of x. Therefore, the roots of this trinomial are either real and equal or imaginary, and its discriminant is less than or equal to zero, i.e.

$$B^2 - AC \leqslant 0.$$

Thus

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 - (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) \leqslant 0,$$

wherefrom also follows that the equality sign is possible only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}.$$

36. Put $b_1 = b_2 = \dots = b_n = 1$ in the inequality of the preceding problem. We then have

$$(a_1 + a_2 + \dots + a_n)^2 \leqslant n(a_1^2 + a_2^2 + \dots + a_n^2).$$

Hence

$$a_1 + a_2 + \dots + a_n \leqslant \sqrt{n(a_1^2 + a_2^2 + \dots + a_n^2)}.$$

37. The result is obtained from the formula of Problem 35 if we put

$$a_1^2 = x_1, \quad a_2^2 = x_2, \quad \dots, \quad a_n^2 = x_n,$$

$$b_1^2 = \frac{1}{x_1}, \quad b_2^2 = \frac{1}{x_2}, \quad \dots, \quad b_n^2 = \frac{1}{x_n}.$$

But we may also use the theorem on the arithmetic mean. Then we have

$$x_1 + x_2 + \dots + x_n \geqslant n\sqrt[n]{x_1x_2 \dots x_n},$$

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \geqslant n\sqrt[n]{\frac{1}{x_1} \cdot \frac{1}{x_2} \dots \frac{1}{x_n}}.$$

Multiplying these inequalities, we get the required result.

38. Let us first prove that

$$p^2 - \frac{2n}{n-1}q \geqslant 0.$$

We have

$$q = x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n,$$

$$0 \leqslant (x_1 - x_2)^2 + (x_1 - x_3)^2 + \ldots + (x_{n-1} - x_n)^2.$$

Consequently

$$(n-1)(x_1^2 + x_2^2 + \ldots + x_n^2) - 2q \geqslant 0.$$

But

$$x_1^2 + x_2^2 + \ldots + x_n^2 = p^2 - 2q,$$

wherefrom we get

$$p^2 - \frac{2n}{n-1} q \geqslant 0.$$

Consider now $n - 1$ quantities (instead of n): $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, eliminating x_i from the quantities under consideration, and put

$$p - x_i = p',$$

$$q - (x_ix_1 + x_ix_2 + \ldots + x_ix_{i-1} + x_ix_{i+1} + \ldots + x_ix_n) = q'.$$

Using the deduced inequality, we may assert that

$$p'^2 - \frac{2(n-1)}{n-2} q' \geqslant 0.$$

But

$$q' = q - x_i(x_1 + x_2 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n) = q - x_i(p - x_i).$$

Therefore

$$(p - x_i)^2 - \frac{2(n-1)}{n-2}(q - px_i + x_i^2) \geqslant 0.$$

Consequently

$$nx_i^2 - 2px_i + 2(n-1)q - (n-2)p^2 \leqslant 0.$$

Consider the trinomial of the second degree

$$nx^2 - 2px + 2(n-1)q - (n-2)p^2$$

and denote its roots by α and β

Solving the quadratic equation, we find

$$\alpha=\frac{p}{n}-\frac{n-1}{n}\sqrt{p^2-\frac{2n}{n-1}q},$$

$$\beta=\frac{p}{n}+\frac{n-1}{n}\sqrt{p^2-\frac{2n}{n-1}q}, \quad (\beta>\alpha).$$

We then have an identity

$$nx_i^2-2px_i+2(n-1)q-(n-2)p^2=$$
$$=n(x_i-\alpha)(x_i-\beta)\leqslant 0,$$

wherefrom follows that x_i lies between α and β, i.e.

$$\alpha<x_i<\beta.$$

39. Let a and b be two real positive numbers. If $p>0$, then $a^p-b^p>0$ for $a>b$; and if $p<0$, then $a^p-b^p<$ <0 for $a>b$. Therefore we may assert the following: $(a^p-b^p)(a^q-b^q)\geqslant 0$ if p and q are of the same sign; $(a^p-b^p)(a^q-b^q)\leqslant 0$ if p and q are of different signs and for any real a and b. Let us first consider the case when p and q are of the same sign. We have

$$a^{p+q}+b^{p+q}\geqslant a^pb^q+a^qb^p,$$
$$a^{p+q}+c^{p+q}\geqslant a^pc^q+a^qc^p,$$
$$\dots\dots\dots\dots$$
$$a^{p+q}+l^{p+q}\geqslant a^pl^q+a^ql^p,$$
$$b^{p+q}+c^{p+q}\geqslant b^pc^q+b^qc^p,$$
$$\dots\dots\dots\dots$$

Adding these inequalities termwise, we get

$$(n-1)(a^{p+q}+b^{p+q}+\dots+l^{p+q})\geqslant\sum a^pb^q,$$

where a and b (in the last sum) attain all the values from the series $a, b, c, \dots, l$. Adding $\sum a^{p+q}$ to both members of this inequality, we get

$$n(a^{p+q}+b^{p+q}+\dots+l^{p+q})\geqslant(a^p+b^p+\dots+$$
$$+l^p)(a^q+b^q+\dots+l^q).$$

The second inequality is obtained just in the same way. From these inequalities we can easily get the results of Problems 36 and 37.

40. 1° Let $\lambda=\frac{m}{n}$, $m>n$. We have

$$\sqrt[m]{\left(1+\alpha\frac{m}{n}\right)\left(1+\alpha\frac{m}{n}\right)\dots\left(1+\alpha\frac{m}{n}\right)\cdot 1\cdot 1\dots 1}<$$

$$<\frac{\left(1+\alpha\frac{m}{n}\right)+\left(1+\alpha\frac{m}{n}\right)+\dots+\left(1+\alpha\frac{m}{n}\right)+m-n}{m}$$

(the factor $1+\alpha\frac{m}{n}$ of the radicand is taken n times. the factor 1 is taken $m-n$ times). Hence

$$\left(1+\alpha\frac{m}{n}\right)^{\frac{n}{m}}<1+\alpha$$

or

$$(1+\alpha)^{\frac{m}{n}}>1+\alpha\frac{m}{n}.$$

2° Put $\lambda=\frac{m}{n}$ and first assume that $m>n$, i.e. $\lambda>1$. We have

$$\sqrt[m]{\left(1-\alpha\frac{m}{n}\right)\left(1-\alpha\frac{m}{n}\right)\dots\left(1-\alpha\frac{m}{n}\right)\cdot 1\cdot 1\dots 1}<$$

$$<\frac{\left(1-\alpha\frac{m}{n}\right)n+m-n}{m}.$$

The factor $1-\alpha\frac{m}{n}$ of the radicand is taken n times, and the factor 1 is taken $m-n$ times. Hence

$$\left(1-\alpha\frac{m}{n}\right)^{\frac{n}{m}}<1-\alpha<\frac{1}{1+\alpha},\qquad 1-\alpha\frac{m}{n}<\frac{1}{(1+\alpha)^{\frac{m}{n}}},$$

$$(1+\alpha)^{\frac{m}{n}}<\frac{1}{1-\alpha\cdot\frac{m}{n}}.$$

Let us assume now that $m<n$. We have

$$\sqrt[n]{(1+\alpha)^m}=\sqrt[n]{(1+\alpha)(1+\alpha)\dots(1+\alpha)\cdot 1\cdot 1\dots 1}<$$

$$<\frac{(1+\alpha)m+n-m}{n}=1+\frac{\alpha m}{n}<\frac{1}{1-\frac{\alpha m}{n}}.$$

And so, in this case also

$$(1+\alpha)^{\frac{m}{n}} < \frac{1}{1-\frac{\alpha m}{n}}.$$

Remember that we assumed $\frac{\alpha m}{n} < 1$.

41. 1° Put in inequality 1° of the preceding problem $\alpha = \frac{1}{n+1}$, $\lambda = \frac{n+1}{n}$. We get

$$\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} > 1+\frac{1}{n}.$$

Hence

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n},$$

i.e. $u_{n+1} > u_n$.

Here is one more proof. Without using the theorem on the arithmetic mean, let us prove that

$$\left(1+\frac{a}{n+1}\right)^{n+1} > \left(1+\frac{a}{n}\right)^{n}$$

if $a > 0$ and n is a positive integer.

Consider the identity

$$1+nx = \frac{1+nx}{1+(n-1)x}\cdot\frac{1+(n-1)x}{1+(n-2)x}\cdots\frac{1+3x}{1+2x}\cdot\frac{1+2x}{1+x}\cdot\frac{1+x}{1}$$
$$(x>0).$$

But

$$\frac{1+(k+1)x}{1+kx} = 1+\frac{x}{1+kx} > 1+\frac{x}{1+nx} = \frac{1+(n+1)x}{1+nx}$$
$$(k=0,\ 1,\ 2,\ \ldots,\ n-1).$$

herefore

$$1+nx > \left[\frac{1+(n+1)x}{1+nx}\right]^n, \quad (1+nx)^{n+1} > [1+(n+1)x]^n.$$

Putting here $x = \frac{a}{n(n+1)}$, we get

$$\left(1+\frac{a}{n+1}\right)^{n+1} > \left(1+\frac{a}{n}\right)^{n}.$$

In particular, at $a=1$, we find

$$\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^{n}.$$

2° We have

$$u_n=\left(1+\frac{1}{n}\right)^{n}=\left[\left(1+\frac{1}{n}\right)^{\frac{n}{k}}\right]^{k}<\left(\frac{1}{1-\frac{1}{n}\cdot\frac{n}{k}}\right)^{k}=\frac{1}{\left(1-\frac{1}{k}\right)^{k}}.$$

Hence

$$u_n<\frac{1}{\left(1-\frac{1}{k}\right)^{k}}$$

for any whole positive k.

If $k=6$, we find

$$\left(1+\frac{1}{n}\right)^{n}<\left(\frac{6}{5}\right)^{6}<3.$$

42. We have

$$\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}=\sqrt[n(n+1)]{\frac{(n+1)^n}{n^{n+1}}}=$$

$$=\sqrt[n(n+1)]{\left(1+\frac{1}{n}\right)^{n}\frac{1}{n}}<\sqrt[n(n+1)]{\frac{3}{n}}$$

(see Problem 41).

But the fraction

$$\frac{3}{n}\leqslant 1 \quad \text{if} \quad n\geqslant 3.$$

Therefore

$$\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}<1 \quad \text{if} \quad n\geqslant 3.$$

43. It is required to prove that

$$\frac{\sqrt[n]{n+1}}{\sqrt[n-1]{n}}<1 \quad (n=2, 3, 4, \ldots).$$

We have

$$\sqrt[n(n-1)]{\frac{(n+1)^{n-1}}{n^n}}=\sqrt[n(n-1)]{\left(1+\frac{1}{n}\right)^{n-1}\frac{1}{n}}=$$

$$=\sqrt[n(n-1)]{\left(1+\frac{1}{n}\right)^{n}\cdot\frac{1}{n+1}}<\sqrt[n(n-1)]{\frac{3}{n+1}}\leqslant 1.$$

44. Let us prove that

$$\log y_i \geqslant a_{i1} \log x_1 + a_{i2} \log x_2 + \ldots + a_{in} \log x_n$$
$$(i = 1, 2, \ldots, n).$$

To this end it suffices to prove that

$$\log (ax + by + cz + \ldots + lu) \geqslant a \log x + b \log y + \ldots + l \log u. \qquad (*)$$

if $a + b + \ldots + l = 1$ and $a, b, \ldots, l$ are rational positive numbers.

Put

$$a = \frac{\alpha}{N}, \quad b = \frac{\beta}{N}, \ldots, l = \frac{\lambda}{N}.$$

Then

$$\alpha + \beta + \ldots + \lambda = N.$$

To prove the inequality (*), it is sufficient to prove that

$$ax + by + cz + \ldots + lu \geqslant x^a y^b \ldots u^l.$$

But we have

$$x^a y^b \ldots u^l = \sqrt[N]{x^\alpha y^\beta \ldots u^\lambda} = \sqrt[N]{x \ldots xy \ldots y \ldots u \ldots u} \leqslant \frac{\alpha x + \beta y + \ldots + \lambda u}{N} = ax + by + \ldots + lu.$$

Thus, it is proved that

$$\log y_i \geqslant a_{i1} \log x_1 + a_{i2} \log x_2 + \ldots + a_{in} \log x_n$$
$$(i = 1, 2, \ldots, n).$$

Hence

$$\sum_{i=1}^{n} \log y_i \geqslant (\log x_1) \sum_{i=1}^{n} a_{i1} + (\log x_2) \sum_{i=1}^{n} a_{i2} + \ldots + (\log x_n) \sum_{i=1}^{n} a_{in}.$$

or

$$\sum_{i=1}^{n} \log y_i \geqslant \log x_1 + \log x_2 + \ldots + \log x_n = \log x_1 x_2 \ldots x_n.$$

Finally

$$y_1y_2 \ldots y_n \geqslant x_1x_2 \ldots x_n.$$

45. Put $\frac{b_i}{a_i} = x_i$ $(i = 1, 2, \ldots, n)$. Then we have to prove the inequality

$$\sqrt[n]{(1+x_1)(1+x_2)\ldots(1+x_n)} \geqslant 1 + \sqrt[n]{x_1x_2\ldots x_n}.$$

The theorem is valid at $n = 1, 2, 3$ (see Problems 21 and 28). Suppose it is true at $n = m$ and let us prove that it also holds at $n = 2m$.

We have

$$\sqrt[2m]{(1+x_1)(1+x_2)\ldots(1+x_{2m-1})(1+x_{2m})} =$$
$$= \sqrt[m]{\sqrt{(1+x_1)(1+x_2)}\cdot\sqrt{(1+x_3)(1+x_4)} \ldots}$$
$$\overline{\ldots \sqrt{(1+x_{2m-1})(1+x_{2m})}} \geqslant$$
$$\geqslant \sqrt[m]{(1+\sqrt{x_1x_2})(1+\sqrt{x_3x_4})\ldots(1+\sqrt{x_{2m-1}x_{2m}})} \geqslant$$
$$\geqslant 1 + \sqrt[m]{\sqrt{x_1x_2}\sqrt{x_3x_4}\ldots\sqrt{x_{2m-1}x_{2m}}} =$$
$$= 1 + \sqrt[2m]{x_1x_2\ldots x_{2m}}.$$

Thus, the theorem is valid for all indices equal to any power of two. Let us now prove that it is true for any whole n. Let $n + q = 2^m$. Then

$$\sqrt[n+q]{(1+x_1)(1+x_2)\ldots(1+x_n)(1+y_1)(1+y_2)\ldots(1+y_q)} \geqslant$$
$$\geqslant 1 + \sqrt[n+q]{x_1x_2\ldots x_ny_1y_2\ldots y_q}.$$

Put

$$1+y_1 = 1+y_2 = \ldots = 1+y_q =$$
$$= \sqrt[n]{(1+x_1)(1+x_2)\ldots(1+x_n)} = Y.$$

We have

$$\sqrt[n+q]{(1+x_1)(1+x_2)\ldots(1+x_n)\cdot Y^q} \geqslant$$
$$\geqslant 1 + \sqrt[n+q]{x_1x_2\ldots x_n(Y-1)^q}$$

But

$$(1+x_1)(1+x_2)\ldots(1+x_n) = Y^n$$

Therefore

$$\sqrt[n+q]{Y^nY^q} \geqslant 1+\sqrt[n+q]{x_1 \dots x_n (Y-1)^q},$$

i.e.

$$Y \geqslant 1+\sqrt[n+q]{x_1x_2 \dots x_n (Y-1)^q}$$

or

$$(Y-1)^{n+q} \geqslant x_1x_2 \dots x_n (Y-1)^q.$$

Hence

$$(Y-1)^n \geqslant x_1x_2 \dots x_n,$$

$$Y-1 \geqslant \sqrt[n]{x_1x_2 \dots x_n}.$$

Finally

$$Y=\sqrt[n]{(1+x_1)(1+x_2) \dots (1+x_n)} \geqslant 1+\sqrt[n]{x_1x_2 \dots x_n},$$

and the theorem is proved.

The equality sign is possible only if $x_1 = x_2 = \dots = x_n = 1$.

46. This theorem, as the previous one, is proved using Cauchy's method. The proposition is valid at $n = 1$; let us first prove that it holds true at $n = 2$, i.e. prove that

$$\left(\frac{x_1+x_2}{2}\right)^k \leqslant \frac{x_1^k+x_2^k}{2} \qquad (*)$$

for any whole positive k. At $k = 1$ the last inequality really takes place. Assuming the validity of this inequality at $k = l$, let us prove its validity at $k = l + 1$. And so, we have (by supposition)

$$\frac{(x_1+x_2)^l}{2^l} \leqslant \frac{x_1^l+x_2^l}{2}.$$

Multiplying both members of this inequality by $\frac{x_1+x_2}{2}$, we find

$$\frac{(x_1+x_2)^{l+1}}{2^{l+1}} \leqslant \frac{(x_1^l+x_2^l)(x_1+x_2)}{4} = \frac{x_1^{l+1}+x_2^{l+1}+x_1x_2^l+x_2x_1^l}{4}.$$

But

$$x_1^lx_2+x_2^lx_1 \leqslant x_1^{l+1}+x_2^{l+1},$$

since

$$x_1^{l+1}+x_2^{l+1}-x_1^l x_2-x_2^l x_1=(x_1-x_2)(x_1^l-x_2^l)\geqslant 0.$$

Therefore

$$\left(\frac{x_1+x_2}{2}\right)^{l+1}\leqslant\frac{x_1^{l+1}+x_2^{l+1}}{2}$$

and the inequality (*) is proved for any whole k. And so, our basic proposition is valid at $n=2$. Let us now prove that if it is true at $n=m$, then it is also true at $n=2m$. Indeed

$$\left(\frac{x_1+x_2+x_3+x_4+\ldots+x_{2m-1}+x_{2m}}{2m}\right)^k=$$

$$=\left(\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}+\ldots+\frac{x_{2m-1}+x_{2m}}{2}}{m}\right)^k\leqslant$$

$$\leqslant\frac{\left(\frac{x_1+x_2}{2}\right)^k+\left(\frac{x_3+x_4}{2}\right)^k+\ldots+\left(\frac{x_{2m-1}+x_{2m}}{2}\right)^k}{m}\leqslant$$

$$\leqslant\frac{\frac{x_1^k+x_2^k}{2}+\frac{x_3^k+x_4^k}{2}+\ldots+\frac{x_{2m-1}^k+x_{2m}^k}{2}}{m}=$$

$$=\frac{x_1^k+x_2^k+x_3^k+x_4^k+\ldots+x_{2m-1}^k+x_{2m}^k}{2m}.$$

Thus, we have established that the theorem is valid at n equal to some power of two. It remains to prove its validity for any whole n. Put $n+p=2^m$.

Then

$$\left(\frac{x_1+x_2+\ldots+x_n+y_1+y_2+\ldots+y_p}{n+p}\right)^k\leqslant$$

$$\leqslant\frac{x_1^k+x_2^k+\ldots+x_n^k+y_1^k+y_2^k+\ldots+y_p^k}{n+p}.$$

Put

$$y_1=y_2=\ldots=y_p=\frac{x_1+x_2+\ldots+x_n}{n}.$$

We have

$$x_1+x_2+\ldots+x_n+y_1+y_2+\ldots+y_p=$$

$$=\frac{(x_1+\ldots+x_n)(n+p)}{n},$$

Hence

$$\left(\frac{x_1+\ldots+x_n}{n}\right)^k \leqslant \frac{x_1^k+\ldots+x_n^k+\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^k p}{n+p}.$$

Finally

$$\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)^k \leqslant \frac{x_1^k+x_2^k+\ldots+x_n^k}{n},$$

and the proposition is completely proved. It is easy to establish that the equality sign is possible only if

$$x_1 = x_2 = \ldots = x_n.$$

47. This proposition is the generalization of the previous theorems (see Problems 30, 45, 46). The proof is carried out in the same way as in the mentioned theorems. Namely, assuming the validity of the theorem at $n = m$, let us prove its validity at $n = 2m$. We have

$$\varphi\left(\frac{t_1+t_2+\ldots+t_{2m}}{2m}\right) = \varphi\left(\frac{\frac{t_1+t_2}{2}+\ldots+\frac{t_{2m-1}+t_{2m}}{2}}{m}\right) \leqslant$$

$$\leqslant \frac{\varphi\left(\frac{t_1+t_2}{2}\right)+\ldots+\varphi\left(\frac{t_{2m-1}+t_{2m}}{2}\right)}{m} <$$

$$< \frac{\frac{\varphi(t_1)+\varphi(t_2)}{2}+\ldots+\frac{\varphi(t_{2m-1})+\varphi(t_{2m})}{2}}{m} =$$

$$= \frac{\varphi(t_1)+\varphi(t_2)+\ldots+\varphi(t_{2m-1})+\varphi(t_{2m})}{2m}$$

(since, by hypothesis, not all of the quantities $t_1, t_2, \ldots, t_{2m}$ are equal to one another, they can be grouped so that, for instance, $t_1 \neq t_2$). Thus, the theorem is valid at $n = 2^m$. Let us put now $n + p = 2^m$. Then

$$\varphi\left(\frac{t_1+t_2+\ldots+t_n+\tau_1+\tau_2+\ldots+\tau_p}{n+p}\right) <$$

$$< \frac{\varphi(t_1)+\ldots+\varphi(t_n)+\varphi(\tau_1)+\ldots+\varphi(\tau_p)}{n+p}$$

(here $t_1, t_2, \ldots, t_n$ are not all equal to one another). Put

$$\tau_1=\tau_2=\ldots=\tau_p=\frac{t_1+t_2+\ldots+t_n}{n},$$

$$\tau_1+\tau_2+\ldots+\tau_p=\frac{t_1+t_2+\ldots+t_n}{n}p.$$

Consequently

$$\varphi\left(\frac{t_1+t_2+\ldots+t_n+\tau_1+\ldots+\tau_p}{n+p}\right)=\varphi\left(\frac{t_1+t_2+\ldots+t_n}{n}\right).$$

On the other hand,

$$\frac{\varphi(t_1)+\ldots+\varphi(t_n)+\varphi(\tau_1)+\ldots+\varphi(\tau_p)}{n+p}=$$

$$=\frac{\varphi(t_1)+\ldots+\varphi(t_n)+p\varphi\left(\frac{t_1+\ldots+t_n}{n}\right)}{n+p}.$$

From the last inequality we get

$$\varphi\left(\frac{t_1+\ldots+t_n}{n}\right)<\frac{\varphi(t_1)+\ldots+\varphi(t_n)}{n}.$$

The above-deduced theorems (see Problems 30, 45, 46) are obtained, as we already mentioned, from this more general proposition. Let us demonstrate this.

1° Let

$$\varphi(t)=-\log(1+t),$$

then

$$\varphi\left(\frac{t_1+t_2}{2}\right)=-\log\left(1+\frac{t_1+t_2}{2}\right).$$

Further

$$\frac{\varphi(t_1)+\varphi(t_2)}{2}=-\frac{\log(1+t_1)+\log(1+t_2)}{2}=$$

$$=-\log\sqrt{(1+t_1)(1+t_2)}.$$

But

$$\sqrt{(1+t_1)(1+t_2)}<\frac{1+t_1+1+t_2}{2}=1+\frac{t_1+t_2}{2}\quad(t_1\neq t_2).$$

Therefore

$$\log\sqrt{1+t_1)(1+t_2)}<\log\left(1+\frac{t_1+t_2}{2}\right)$$

(the base of the logarithms being greater than unity) and

$$-\log \sqrt{(1+t_1)(1+t_2)} > -\log\left(1+\frac{t_1+t_2}{2}\right).$$

Thus, the function

$$\varphi(t) = -\log(1+t)$$

really possesses the following property

$$\varphi\left(\frac{t_1+t_2}{2}\right) < \frac{\varphi(t_1)+\varphi(t_2)}{2},$$

and therefore it must be

$$\varphi\left(\frac{t_1+t_2+\ldots+t_n}{n}\right) < \frac{\varphi(t_1)+\varphi(t_2)+\ldots+\varphi(t_n)}{n},$$

i.e.

$$-\log\left(1+\frac{t_1+t_2+\ldots+t_n}{n}\right) <$$
$$< -\frac{\log(1+t_1)+\log(1+t_2)+\ldots+\log(1+t_n)}{n},$$

$$\log \sqrt[n]{(1+t_1)(1+t_2)\ldots(1+t_n)} <$$
$$< \log\left(1+\frac{t_1+\ldots+t_n}{n}\right).$$

Further

$$\sqrt[n]{(1+t_1)(1+t_2)\ldots(1+t_n)} <$$
$$< 1+\frac{t_1+\ldots+t_n}{n} =$$
$$= \frac{(1+t_1)+(1+t_2)+\ldots+(1+t_n)}{n}.$$

Putting $1+t_i = x_i$, we finally get

$$\sqrt[n]{x_1x_2\ldots x_n} < \frac{x_1+x_2+\ldots+x_n}{n}.$$

Obviously, if we assume the possibility $x_1 = x_2 = \ldots = x_n$, then it will be

$$\sqrt[n]{x_1x_2\ldots x_n} \leqslant \frac{x_1+x_2+\ldots+x_n}{n}.$$

2° If we put

$$\varphi(t) = t^k,$$

then

$$\varphi\left(\frac{t_1+t_2}{2}\right)=\left(\frac{t_1+t_2}{2}\right)^k.$$

Assuming that the inequality

$$\left(\frac{t_1+t_2}{2}\right)^k<\frac{t_1^k+t_2^k}{2}$$

holds true, we get the result of Problem 46.

3° Put

$$\varphi(t)=\log(1+e^t)$$

(the logarithm is taken to the base $e>1$).

Then

$$\varphi\left(\frac{t_1+t_2}{2}\right)=\log\left(1+e^{\frac{t_1+t_2}{2}}\right),$$

$$\frac{\varphi(t_1)+\varphi(t_2)}{2}=\log\sqrt{(1+e^{t_1})(1+e^{t_2})}.$$

Since

$$\sqrt{(1+e^{t_1})(1+e^{t_2})}>1+e^{\frac{t_1+t_2}{2}},$$

fulfilled for the function $\varphi(t)$ is the inequality

$$\varphi\left(\frac{t_1+t_2}{2}\right)<\frac{\varphi(t_1)+\varphi(t_2)}{2}\quad(t_1\neq t_2).$$

Therefore

$$\varphi\left(\frac{t_1+t_2+\ldots+t_n}{n}\right)<\frac{\varphi(t_1)+\ldots+\varphi(t_n)}{n},$$

i.e.

$$\log\left(1+e^{\frac{t_1+t_2+\ldots+t_n}{n}}\right)<\frac{\log(1+e^{t_1})+\ldots+\log(1+e^{t_n})}{n},$$

$$1+e^{\frac{t_1+t_2+\ldots+t_n}{n}}<\sqrt[n]{(1+e^{t_1})\ldots(1+e^{t_n})}.$$

Put

$$e^t=\lambda,\quad t=\log_e\lambda.$$

Then

$$\sqrt[n]{(1+e^{t_1})\ldots(1+e^{t_n})}=$$

$$=\sqrt[n]{(1+\lambda_1)(1+\lambda_2)\ldots(1+\lambda_n)}>$$

$$>1+e^{\frac{\log\lambda_1+\ldots+\log\lambda_n}{n}}$$

Finally

$$\sqrt[n]{(1+\lambda_1)(1+\lambda_2)\dots(1+\lambda_n)} > 1+\sqrt[n]{\lambda_1\lambda_2\dots\lambda_n}.$$

48. Let $t_1, t_2, \dots, t_n$ be contained in the interval between 0 and π.

$$(0 < t_i < \pi).$$

Let us prove that

$$-\sin\frac{t_1+t_2+\dots+t_n}{n} < -\frac{\sin t_1+\sin t_2+\dots+\sin t_n}{n}.$$

For this purpose it suffices to prove that (see Problem 47)

$$-\sin\frac{t_1+t_2}{2} < -\frac{\sin t_1+\sin t_2}{2}.$$

Indeed

$$\sin\frac{t_1+t_2}{2}-\frac{\sin t_1+\sin t_2}{2}=\sin\frac{t_1+t_2}{2}-$$
$$-\sin\frac{t_1+t_2}{2}\cos\frac{t_1-t_2}{2}=$$
$$=\sin\frac{t_1+t_2}{2}\cdot 2\sin^2\frac{t_1-t_2}{4}>0$$

(in our case $\varphi(t)=-\sin t$).

Thus

$$\frac{\sin t_1+\sin t_2+\dots+\sin t_n}{n} < \sin\frac{t_1+t_2+\dots+t_n}{n}$$

(if $0<t_i<\pi$).

Therefore if $a_1+a_2+\dots+a_n=\pi$, then

$$\sin a_1+\sin a_2+\dots+\sin a_n < n\sin\frac{\pi}{n}$$

if $a_1, a_2, \dots, a_n$ are not equal to one another.

On the other hand, if

$$a_1=a_2=\dots=a_n=\frac{\pi}{n},$$

then the sum

$$\sin a_1+\dots+\sin a_n$$

becomes equal to

$$n\sin\frac{\pi}{n}.$$

Thus, indeed, the greatest value of the sum

$$\sin a_1 + \sin a_2 + \ldots + \sin a_n$$

will be

$$n \sin \frac{\pi}{n},$$

provided

$$a_1 + a_2 + \ldots + a_n = \pi \quad (a_i > 0);$$

and this greatest value is attained at

$$a_1 = a_2 = \ldots = a_n = \frac{\pi}{n}.$$

49. Let us prove that the difference

$$\frac{x^p - 1}{p} - \frac{x^q - 1}{q}$$

(if $x \neq 1$ and $p > q$) exceeds zero. To this end it is sufficient to prove that

$$\Delta = q(x^p - 1) - p(x^q - 1) > 0.$$

First let us assume that $x > 1$. We have

$$\Delta = q(x^p - 1) - p(x^q - 1) = (x - 1)\{q(x^{p-1} + x^{p-2} + \ldots + x + 1) - p(x^{q-1} + x^{q-2} + \ldots + x + 1)\} = (x - 1)\{q(x^{p-1} + x^{p-2} + \ldots + x^q) - (p - q)(x^{q-1} + x^{q-2} + \ldots + x + 1)\}.$$

If $x > 1$, then

$$x^{p-1} + x^{p-2} + \ldots + x^q > (p - q)x^q.$$

Therefore

$$\Delta = q(x^p - 1) - p(x^q - 1) > (x - 1)\{q(p - q)x^q - (p - q)qx^{q-1}\} = qx^{q-1}(p - q)(x - 1)^2 > 0.$$

Thus, if $x > 1$, the theorem is proved. Now let us assume that $x < 1$. In this case we have

$$x^{p-1} + x^{p-2} + \ldots + x^q < (p - q)x^q,$$

$$x^{q-1} + x^{q-2} + \ldots + x + 1 > qx^{q-1},$$

$$q(x^{p-1} + \ldots + x^q) - (p - q)(x^{q-1} + \ldots + x + 1) < (p - q)qx^q - q(p - q)x^{q-1} = q(p - q)x^{q-1}(x - 1).$$

Consequently

$$\Delta > q(p-q)\,x^{q-1}(x-1)^2 > 0.$$

However, this proposition can be proved proceeding from the theorem on the arithmetic mean. We have the following inequality (see Problem 40)

$$(1+\alpha)^{\lambda} > 1+\alpha\lambda$$

($\lambda > 1$, rational, $\alpha > 0$, real).

Likewise we can deduce the following inequality

$$(1-\alpha)^{\lambda} > 1-\alpha\lambda$$

if $0<\alpha<1$; $\lambda>1$, rational. Using these inequalities, we shall prove that

$$\frac{x^p-1}{p} > \frac{x^q-1}{q}$$

if $p>q$ $(x \neq 1)$.

Put $x^q=\xi$, $\frac{p}{q}=\lambda$. Then we have to prove

$$\xi^{\lambda}-1 > \lambda(\xi-1)$$

or

$$\xi^{\lambda}-1-\lambda(\xi-1) > 0.$$

First suppose $x>1$, $\xi>1$. Put $\xi=1+\alpha$. We then have

$$\xi^{\lambda}-1-\lambda(\xi-1) = (1+\alpha)^{\lambda}-1-\lambda\alpha > 0.$$

If $x<1$, then $\xi<1$. In this case we put

$$\xi = 1-\alpha \quad (0<\alpha<1).$$

We find easily

$$\xi^{\lambda}-1-\lambda(\xi-1) = (1-\alpha)^{\lambda}-1-\lambda(-\alpha) > 0.$$

50. Let us first assume that $m>1$. Put $m=\frac{p}{q}$ ($p>q$, positive integer). We then have (see Problem 49)

$$\frac{\xi^p-1}{p} > \frac{\xi^q-1}{q} \quad (\xi \neq 1).$$

Putting $\xi^q=x$, $\xi=x^{\frac{1}{q}}$, we get

$$x^m-1 > m(x-1).$$

Replacing in this inequality x by $\frac{1}{x}$, we find

$$\frac{1}{x^m}-1>m\left(\frac{1}{x}-1\right).$$

Multiplying both members of this inequality by $-x^m$, we get

$$x^m-1<mx^{m-1}(x-1).$$

Thus, if $m>1$, then

$$mx^{m-1}(x-1)>x^m-1>m(x-1). \qquad (1)$$

Let us assume now that $0<m<1$. Putting $\xi^q=x$, $\frac{q}{p}=m$, we find

$$x^{\frac{1}{m}}-1>\frac{1}{m}(x-1).$$

Replacing here x by x^m, we find

$$x^m-1<m(x-1).$$

Replacing in the last inequality x by $\frac{1}{x}$, and performing all necessary transformations, we find

$$mx^{m-1}(x-1)<x^m-1<m(x-1) \quad (0<m<1). \qquad (2)$$

Let us now consider negative values of m. Put $m=-n$, where $n>0$, rational. Let us first prove that if m is negative, then

$$x^m-1>m(x-1).$$

Since $n>0$, it follows that $n+1>1$ and we may make use of inequalities (1). Namely, we have

$$x^{n+1}-1<(n+1)x^n(x-1).$$

Hence

$$nx^n(x-1)>x^n-1.$$

Replacing here n by $-m$, we find

$$-mx^{-m}(x-1)>x^{-m}-1.$$

Multiplying both members of this inequality by $-x^m$, we get

$$x^m-1>m(x-1).$$

And if we replace here x by $\frac{1}{x}$, then we find

$$x^m - 1 < mx^{m-1}(x-1).$$

Thus, indeed

$$mx^{m-1}(x-1) < x^m - 1 < m(x-1),$$

if $0 < m < 1$,

$$m(x-1) < x^m - 1 < mx^{m-1}(x-1)$$

if m is any rational number not lying in the interval between 0 and 1, and x is any real positive number not equal to unity.

51. The inequalities of this problem follow immediately from the results of the preceding problem.

52. Put

$$x_i^p = y_i, \quad \frac{q}{p} = m.$$

Then the inequality is rewritten as follows

$$\left(\frac{y_1 + y_2 + \ldots + y_n}{n}\right)^m \leqslant \frac{y_1^m + y_2^m + \ldots + y_n^m}{n},$$

where $m \geqslant 1$, rational. Using the results of Problem 47, it is sufficient to prove that

$$\left(\frac{t_1 + t_2}{2}\right)^m \leqslant \frac{t_1^m + t_2^m}{2}$$

for any rational $m > 1$ and for any real positive t_1 and t_2. In other words, it is sufficient to prove that

$$\left(\frac{2t_1}{t_1 + t_2}\right)^m + \left(\frac{2t_2}{t_1 + t_2}\right)^m \geqslant 2. \qquad (1)$$

Let us make use of the results of Problem 51

$$(1 + x)^m \geqslant 1 + mx$$

if $m > 1$ is rational and $1 + x > 0$. We have two inequalities

$$\left(\frac{2t_1}{t_1 + t_2}\right)^m \geqslant 1 + m\left(\frac{2t_1}{t_1 + t_2} - 1\right),$$

$$\left(\frac{2t_2}{t_1 + t_2}\right)^m \geqslant 1 + m\left(\frac{2t_2}{t_1 + t_2} - 1\right).$$

Adding them, we get inequality (1) which is the required result. The solution to our problem can be obtained immediately from the inequalities of Problem 51. Let us show that, using this method, we can deduce even a more general inequality. So let us prove that

$$\left(\frac{y_1+y_2+\ldots+y_n}{n}\right)^{\lambda} \leqslant \frac{y_1^{\lambda}+y_2^{\lambda}+\ldots+y_n^{\lambda}}{n}$$

if λ is a rational number not lying in the interval between zero and unity and

$$\left(\frac{y_1+y_2+\ldots+y_n}{n}\right)^{\lambda} \geqslant \frac{y_1^{\lambda}+y_2^{\lambda}+\ldots+y_n^{\lambda}}{n}$$

if $0<\lambda<1$. To prove the first inequality it is sufficient to prove that

$$\left(\frac{ny_1}{y_1+y_2+\ldots+y_n}\right)^{\lambda}+\left(\frac{ny_2}{y_1+y_2+\ldots+y_n}\right)^{\lambda}+\ldots+$$
$$+\left(\frac{ny_n}{y_1+\ldots+y_n}\right)^{\lambda} \geqslant n. \quad (2)$$

But we have (see Problem 51)

$$\left(\frac{ny_i}{y_1+y_2+\ldots+y_n}\right)^{\lambda} \geqslant 1+\lambda\left(\frac{ny_i}{y_1+y_2+\ldots+y_n}-1\right).$$

Putting here $i=1, 2, \ldots, n$ and adding the inequalities thus obtained, we actually get inequality (2). We proceed quite analogously for the case $0<\lambda<1$.

53. Put

$$x_1+x_2+\ldots+x_n=p, \quad x_1^2+x_2^2+\ldots+x_n^2=p'.$$

We have

$$(x-x_1)^2+(x-x_2)^2+\ldots+(x-x_n)^2=$$
$$=nx^2-2px+p'=n\left[x^2-\frac{2p}{n}x+\frac{p'}{n}\right]=$$
$$=n\left[\left(x-\frac{p}{n}\right)^2+\frac{p'}{n}-\frac{p^2}{n^2}\right].$$

Our expression can attain the least value only simultaneously with $\left(x-\frac{p}{n}\right)^2$ $\left(\text{since the quantity } \frac{p'}{n}-\frac{p^2}{n^2} \text{ is independent of } x\right)$. But $\left(x-\frac{p}{n}\right)^2$ cannot be negative, therefore its least value will be equal to zero. Hence

$$x = \frac{p}{n} = \frac{x_1 + \ldots + x_n}{n}.$$

Thus, the sum

$$(x - x_1)^2 + (x - x_2)^2 + \ldots + (x - x_n)^2$$

attains the least value at

$$x = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

54. Put

$$x_1^2 + x_2^2 + \ldots + x_n^2 = S_2.$$

Then

$$(x_1 - x_2)^2 + (x_1 - x_3)^2 + \ldots + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x_n)^2 = (n-1) S_2 - 2q,$$

where

$$q = x_1x_2 + x_1x_3 + \ldots + x_1x_n + x_2x_3 + \ldots + x_{n-1}x_n.$$

Further

$$(x_1 + x_2 + \ldots + x_n)^2 = S_2 + 2q.$$

And so

$$(n-1) S_2 = 2q + \sum_{j>i} (x_i - x_j)^2,$$

$$C^2 = S_2 + 2q,$$

wherefrom we find

$$nS_2 = C^2 + \sum_{j>i} (x_i - x_j)^2.$$

The last equality shows that S_2 takes the least value when the least value is attained by $\sum_{j>i} (x_i - x_j)^2$. The least value of this sum is equal to zero and is attained at

$$x_1 = x_2 = \ldots = x_n.$$

But since

$$x_1 + x_2 + \ldots + x_n = C,$$

it follows that

$$x_1^2 + \ldots + x_n^2$$

takes on the least value at

$$x_1 = x_2 = \dots = x_n = \frac{C}{n}.$$

55. First let us assume that λ does not lie in the interval between 0 and 1. Then the following inequality takes place

$$\frac{x_1^\lambda + x_2^\lambda + \dots + x_n^\lambda}{n} \geqslant \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^\lambda,$$

the equality sign (as it is easy to find out) occurring only if

$$x_1 = x_2 = \dots = x_n.$$

If it is given that

$$x_1 + x_2 + \dots + x_n = C,$$

then at all values of $x_1, x_2, \dots, x_n$, we have

$$x_1^\lambda + x_2^\lambda + \dots + x_n^\lambda \geqslant n\left(\frac{C}{n}\right)^\lambda,$$

wherefrom it is seen that the least value of the expression

$$x_1^\lambda + x_2^\lambda + \dots + x_n^\lambda$$

is $n\left(\frac{C}{n}\right)^\lambda$ which is reached at $x_1 = x_2 = \dots = x_n = \frac{C}{n}$. But if $0 < \lambda < 1$, then the following inequality takes place

$$\frac{x_1^\lambda + x_2^\lambda + \dots + x_n^\lambda}{n} \leqslant \left(\frac{x_1 + \dots + x_n}{n}\right)^\lambda.$$

Then at

$$x_1 = x_2 = \dots = x_n$$

we obtain the least value of the quantity

$$x_1^\lambda + x_2^\lambda + \dots + x_n^\lambda.$$

56. We have the inequality (see problem 30)

$$\sqrt[n]{x_1 x_2 \dots x_n} \leqslant \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{C}{n}.$$

Hence

$$x_1 x_2 \dots x_n \leqslant \left(\frac{C}{n}\right)^n.$$

Thus, the product $x_1x_2 \ldots x_n$ does not exceed $\left(\frac{C}{n}\right)^n$ and reaches it only at $x_1 = x_2 = \ldots = x_n = \frac{C}{n}$ (see Problem 30). And so the greatest value is attained by the product $x_1x_2 \ldots x_n$ when

$$x_1 = x_2 = \ldots = x_n = \frac{C}{n}.$$

57. We have

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1x_2 \ldots x_n}.$$

Consequently

$$x_1 + x_2 + \ldots + x_n \geqslant n\sqrt[n]{C}.$$

The equality sign being possible if $x_1 = x_2 = \ldots = x_n$. Hence, it is clear that the sum $x_1 + x_2 + \ldots + x_n$ attains the least value if

$$x_1 = x_2 = \ldots = x_n = \sqrt[n]{C}.$$

58. First let us assume that μ_i $(i = 1, 2, \ldots, n)$ are whole numbers. We have

$$\sqrt[\mu_1+\mu_2+\mu_3+\ldots+\mu_n]{\left(\frac{x_1}{\mu_1}\right)^{\mu_1}\left(\frac{x_2}{\mu_2}\right)^{\mu_2}\cdots\left(\frac{x_n}{\mu_n}\right)^{\mu_n}} =$$

$$= \sqrt[\mu_1+\ldots+\mu_n]{\frac{x_1}{\mu_1}\cdot\frac{x_1}{\mu_1}\cdots\frac{x_1}{\mu_1}\cdot\frac{x_2}{\mu_2}\cdots\frac{x_2}{\mu_2}\cdots\frac{x_n}{\mu_n}\cdots\frac{x_n}{\mu_n}} \leqslant$$

$$\leqslant \frac{\mu_1\frac{x_1}{\mu_1} + \mu_2\frac{x_2}{\mu_2} + \ldots + \mu_n\frac{x_n}{\mu_n}}{\mu_1 + \mu_2 + \ldots + \mu_n} = \frac{C}{\mu_1 + \ldots + \mu_n}.$$

Consequently

$$x_1^{\mu_1}x_2^{\mu_2}\ldots x_n^{\mu_n} \leqslant \left(\frac{C}{\mu_1+\ldots+\mu_n}\right)^{\mu_1+\mu_2+\ldots+\mu_n}\cdot\mu_1^{\mu_1}\cdot\mu_2^{\mu_2}\ldots\mu_n^{\mu_n},$$

and the equality sign is obtained only if

$$\frac{x_1}{\mu_1} = \frac{x_2}{\mu_2} = \ldots = \frac{x_n}{\mu_n}.$$

Let now μ_i be fractions. Reducing them to a common denominator, we put

$$\mu_i = \frac{\lambda_i}{\mu},$$

where λ_i and μ are positive integers.

Since

$$x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n} = \sqrt[\mu]{x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}},$$

the greatest value is reached by the product $x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$ simultaneously with the product $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$, where λ_i are integers. As follows from the above-proved, it happens if and only if

$$\frac{x_1}{\lambda_1} = \frac{x_2}{\lambda_2} = \ldots = \frac{x_n}{\lambda_n}.$$

Dividing the denominators by μ, we get

$$\frac{x_1}{\mu_1} = \frac{x_2}{\mu_2} = \ldots = \frac{x_n}{\mu_n}.$$

Thus, if $x_i > 0$ and $x_1 + x_2 + \ldots + x_n = C$, then the product $x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$ ($\mu_i > 0$, rational) attains the greatest value if and only if

$$\frac{x_1}{\mu_1} = \frac{x_2}{\mu_2} = \ldots = \frac{x_n}{\mu_n}.$$

59. We have

$$\sqrt[n]{a_1 x_1 \cdot a_2 x_2 \ldots a_n x_n} \leqslant \frac{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n}{n} = \frac{C}{n},$$

wherefrom it follows that the product

$$a_1 x_1 \cdot a_2 x_2 \ldots a_n x_n$$

reaches the greatest value only if

$$a_1 x_1 = a_2 x_2 = \ldots = a_n x_n.$$

But since

$$a_1 x_1 \cdot a_2 x_2 \ldots a_n x_n = (a_1 a_2 \ldots a_n)(x_1 x_2 \ldots x_n),$$

the product $x_1 x_2 \ldots x_n$ indeed reaches the greatest value if and only if

$$a_1 x_1 = a_2 x_2 = \ldots = a_n x_n = \frac{C}{n}.$$

60. Put

$$a_i x_i^{\lambda_i} = y_i \quad (i = 1, 2, \ldots, n).$$

Then

$$x_i = \left(\frac{y_i}{a_i}\right)^{\frac{1}{\lambda_i}}$$

and

$$y_1 + y_2 + \ldots + y_n = C.$$

Further

$$x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n} = \left(\frac{y_1}{a_1}\right)^{\frac{\mu_1}{\lambda_1}} \left(\frac{y_2}{a_2}\right)^{\frac{\mu_2}{\lambda_2}} \ldots \left(\frac{y_n}{a_n}\right)^{\frac{\mu_n}{\lambda_n}}.$$

The problem is reduced to finding out when the product

$$y_1^{\frac{\mu_1}{\lambda_1}} \cdot y_2^{\frac{\mu_2}{\lambda_2}} \ldots y_n^{\frac{\mu_n}{\lambda_n}}$$

takes on the greatest value if $y_1 + y_2 + \ldots + y_n = C$. Referring to the results of Problem 58, we see that it will take place if

$$\frac{y_1}{\frac{\mu_1}{\lambda_1}} = \frac{y_2}{\frac{\mu_2}{\lambda_2}} = \ldots = \frac{y_n}{\frac{\mu_n}{\lambda_n}}.$$

Thus, if

$$a_1 x_1^{\lambda_1} a_2 x_2^{\lambda_2} + \ldots + a_n x_n^{\lambda_n} = C,$$

then the product

$$x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$$

reaches the greatest value provided

$$\frac{\lambda_1 a_1 x_1^{\lambda_1}}{\mu_1} = \frac{\lambda_2 a_2 x_2^{\lambda_2}}{\mu_2} = \ldots = \frac{\lambda_n a_n x_n^{\lambda_n}}{\mu_n}.$$

61. Put

$$a_1 x_1^{\mu_1} = y_1, \quad a_2 x_2^{\mu_2} = y_2, \quad \ldots, \quad a_n x_n^{\mu_n} = y_n.$$

Hence

$$x_1 = \left(\frac{y_1}{a_1}\right)^{\frac{1}{\mu_1}}, \quad x_2 = \left(\frac{y_2}{a_2}\right)^{\frac{1}{\mu_2}}, \quad \ldots, \quad x_n = \left(\frac{y_n}{a_n}\right)^{\frac{1}{\mu_n}},$$

and the problem is reduced to the following: under what condition does the sum

$$y_1 + y_2 + \dots + y_n$$

attain the least value if

$$y_1^{\frac{\lambda_1}{\mu_1}} \cdot y_2^{\frac{\lambda_2}{\mu_2}} \dots y_n^{\frac{\lambda_n}{\mu_n}} = C_1,$$

where C_1 is a new constant?

Since $\frac{\lambda_1}{\mu_1}, \dots, \frac{\lambda_n}{\mu_n}$ are rational, we put

$$\frac{\lambda_1}{\mu_1} = \frac{\alpha_1}{N}, \quad \frac{\lambda_2}{\mu_2} = \frac{\alpha_2}{N}, \quad \dots, \quad \frac{\lambda_n}{\mu_n} = \frac{\alpha_n}{N}.$$

Then the problem will read as follows: find out when $y_1 + y_2 + \dots + y_n$ attains the least value if

$$y_1^{\alpha_1} y_2^{\alpha_2} \dots y_n^{\alpha_n} = C_2 \quad (\alpha_i \text{ positive integers}).$$

Finally, we put

$$y_1 = \alpha_1 u_1, \quad y_2 = \alpha_2 u_2, \quad \dots, \quad y_n = \alpha_n u_n$$

and obtain the following problem: under what conditions does

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$$

attain the least value if

$$u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n} = C_3.$$

But

$$\frac{\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n}{\alpha_1 + \alpha_2 + \dots + \alpha_n} \geqslant$$

$$\geqslant \sqrt[\alpha_1+\alpha_2+\dots+\alpha_n]{u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n}} = \sqrt[\alpha_1+\alpha_2+\dots+\alpha_n]{C_3}.$$

Hence $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ attains the least value when

$$u_1 = u_2 = \dots = u_n.$$

Thus, if

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} = C,$$

then

$$a_1 x_1^{\mu_1} + a_2 x_2^{\mu_2} + \dots + a_n x_n^{\mu_n}$$

attains the least value provided

$$\frac{\dfrac{x_1^{\mu_1}}{\lambda_1}}{a_1\mu_1}=\frac{\dfrac{x_2^{\mu_2}}{\lambda_2}}{a_2\mu_2}=\ldots=\frac{\dfrac{x_n^{\mu_n}}{\lambda_n}}{a_n\mu_n}.$$

62. Applying the Lagrange formula (see Problem 5, Sec. 1), we have

$$(x^2+y^2+z^2+\ldots+t^2)(a^2+b^2+c^2+\ldots+k^2)=$$
$$=(ax+by+\ldots+kt)^2+(xb-ya)^2+$$
$$+(xc-za)^2+\ldots.$$

Since

$$a^2+b^2+c^2+\ldots+k^2$$

is constant and

$$ax+by+\ldots+kt=A$$

(by hypothesis) and, consequently, also constant, it follows that the sum

$$x^2+y^2+z^2+\ldots+t^2$$

attains the least value simultaneously with the sum

$$(xb-ya)^2+(xc-za)^2+\ldots.$$

But the least value of the latter sum is zero which is reached when

$$xb-ya=0,\quad xc-za=0,\ \ldots,$$

i.e. when

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\ldots=\frac{t}{k}.$$

Let us put this general ratio equal to λ so that

$$x=a\lambda,\quad y=b\lambda,\quad z=c\lambda,\ \ldots,\ t=k\lambda.$$

Substituting these values for $x, y, z, \ldots, t$ into the equality

$$ax+by+\ldots+kt=A,$$

we find

$$\lambda=\frac{A}{a^2+b^2+\ldots+k^2},$$

and, consequently, the required values of $x, y, \ldots, t$ at which the expression $x^2+y^2+\ldots+t^2$ takes on the least value will be

$$x = \frac{aA}{a^2 + b^2 + \ldots + k^2},$$

$$y = \frac{bA}{a^2 + b^2 + \ldots + k^2}, \quad \ldots, \quad t = \frac{kA}{a^2 + b^2 + \ldots + k^2}.$$

63. We have

$$u = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F,$$

where

$$A = a_1^2 + a_2^2 + \ldots + a_n^2, \qquad B = a_1b_1 + a_2b_2 + \ldots + a_nb_n,$$
$$C = b_1^2 + b_2^2 + \ldots + b_n^2, \qquad D = a_1c_1 + a_2c_2 + \ldots + a_nc_n,$$
$$E = b_1c_1 + b_2c_2 + \ldots + b_nc_n, \quad F = c_1^2 + c_2^2 + \ldots + c_n^2.$$

Put

$$x = x' + \alpha, \quad y = y' + \beta.$$

We then obtain

$$u = A(x' + \alpha)^2 + 2B(x' + \alpha)(y' + \beta) + C(y' + \beta)^2 + \\ + 2D(x' + \alpha) + 2E(y' + \beta) + F.$$

Expanding this expression in powers of x' and y', we get

$$u = Ax'^2 + 2Bx'y' + Cy'^2 + 2(A\alpha + B\beta + D)x' + \\ + 2(B\alpha + C\beta + E)y' + F'.$$

Now let us choose α and β so that the coefficients of x' and y' in the last expansion equal zero. To this end it is only necessary to choose α and β as the solutions of the following system

$$A\alpha + B\beta + D = 0,$$
$$B\alpha + C\beta + E = 0.$$

Then we have

$$u = Ax'^2 + 2Bx'y' + Cy'^2 + F'.$$

Further

$$u = \frac{1}{A}\{A^2x'^2 + 2BAx'y' + ACy'^2\} + F' = \\ = \frac{1}{A}\{(Ax' + By')^2 + (AC - B^2)y'^2\} + F'.$$

But

$$AC - B^2 = (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) - \\ - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \geqslant 0, \quad A > 0.$$

Therefore, u attains the least value when

$$Ax' + By' = 0 \text{ and } y' = 0.$$

Hence

$$x' = y' = 0 \text{ and } x = \alpha, \; y = \beta.$$

And so, the values of x and y at which u attains the least value are obtained as the solution of the following system of equations

$$Ax + By + D = 0, \quad Bx + Cy + E = 0.$$

However, this result can be obtained in a somewhat different way.

Put

$$a_1x + b_1y + c_1 = X_1, \quad a_2x + b_2y + c_2 = X_2, \ldots, \\ a_nx + b_ny + c_n = X_n.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be some constants satisfying the following conditions

$$\begin{aligned} a_1\lambda_1 + a_2\lambda_2 + \ldots + a_n\lambda_n &= 0, \\ b_1\lambda_1 + b_2\lambda_2 + \ldots + b_n\lambda_n &= 0, \\ c_1\lambda_1 + c_2\lambda_2 + \ldots + c_n\lambda_n &= k, \end{aligned} \tag{*}$$

where k is an arbitrary number.

We then have

$$\lambda_1X_1 + \lambda_2X_2 + \ldots + \lambda_nX_n = k$$

and hence, we have to find the least value of the expression

$$X_1^2 + X_2^2 + \ldots + X_n^2,$$

provided

$$\lambda_1X_1 + \lambda_2X_2 + \ldots + \lambda_nX_n = k \text{ (constant)}.$$

From the result of Problem 62 we have that the least value is obtained if

$$\frac{X_1}{\lambda_1} = \frac{X_2}{\lambda_2} = \ldots = \frac{X_n}{\lambda_n}.$$

Or

$$\lambda_1 = X_1\mu, \quad \lambda_2 = X_2\mu, \ \ldots, \ \lambda_n = X_n\mu.$$

Substituting them into the first two equalities (*), we find

$$a_1X_1 + a_2X_2 + \ldots + a_nX_n = 0,$$
$$b_1X_1 + b_2X_2 + \ldots + b_nX_n = 0.$$

Hence we get the system obtained by the preceding method of solution.

64. As is known, there exists the following identity (see Problem 77, Sec. 6)

$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} + f(x_1)\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} + \ldots + f(x_n)\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})},$$

where $f(x)$ is any polynomial of degree n.

Equating the coefficients at x^n in both members of this equality, we find

$$1 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} + \ldots + \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}.$$

Let M denote the greatest one of the quantities

$$|f(x_0)|, \quad |f(x_1)|, \quad \ldots, \quad |f(x_n)|.$$

Then

$$1 \leqslant M\left\{\frac{1}{|(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)|} + \frac{1}{|(x_1-x_0)\ldots(x_1-x_n)|} + \ldots + \frac{1}{|(x_n-x_0)\ldots(x_n-x_{n-1})|}\right\}.$$

As is easily seen, by virtue of our conditions we have

$$|(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)| \geqslant k!\,(n-k)!.$$

Therefore

$$\frac{1}{|(x_k-x_0)(x_k-x_1)\dots(x_k-x_n)|}\leqslant\frac{1}{k!\,(n-k)!}.$$

Consequently

$$1\leqslant M\sum_{k=0}^{n}\frac{1}{k!\,(n-k)!}=\frac{M}{n!}\sum_{k=0}^{n}C_n^k=M\frac{2^n}{n!}.$$

Finally

$$M\geqslant\frac{n!}{2^n}.$$

65. Since $\sin^2 x+\cos^2 x=1$, i.e. the sum of the two quantities $\sin^2 x$ and $\cos^2 x$ is constant, their product $\sin^2 x\cdot\cos^2 x$ reaches the greatest value when these quantities are equal to each other. It happens at $x=\frac{\pi}{4}$. However, the same is easily seen from the identity

$$\sin x\cdot\cos x=\frac{1}{2}\sin 2x.$$

66. It is known that if

$$x+y+z=\frac{\pi}{2},$$

then

$$\tan x\tan y+\tan x\tan z+\tan y\tan z=1$$

(see Problem 40, 4°, Sec. 2). Thus, the sum of the three quantities

$$\tan x\tan y,\quad \tan x\tan z,\quad \tan y\tan z$$

is constant. Therefore, the product of these quantities

$$\tan^2 x\tan^2 y\tan^2 z$$

reaches the greatest value if

$$\tan x\tan y=\tan x\tan z=\tan y\tan z,$$

i.e. if

$$\tan x=\tan y=\tan z$$

and consequently at

$$x=y=z=\frac{\pi}{6}.$$

67. We have

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}+\frac{1}{3n+1}=\left(\frac{1}{n+1}+\frac{1}{3n+1}\right)+$$
$$+\left(\frac{1}{n+2}+\frac{1}{3n}\right)+\ldots+\frac{1}{2n+1}=\frac{4n+2}{(n+1)(3n+1)}+$$
$$+\frac{4n+2}{(n+2)\,3n}+\ldots+\frac{4n+2}{2\,(2n+1)^2}>$$
$$>(4n+2)\left\{\frac{n}{(2n+1)^2}+\frac{1}{2\,(2n+1)^2}\right\}=1.$$

68. Put

$$a=\alpha^2.$$

It is required to prove that

$$\alpha^{2n}-1\geqslant n\,(\alpha^{n+1}-\alpha^{n-1}).$$

Or, which is the same,

$$\alpha^{2n}-1\geqslant n\alpha^{n-1}(\alpha^2-1),\quad \frac{\alpha^{2n}-1}{\alpha^2-1}\geqslant n\alpha^{n-1}.$$

But

$$\frac{\alpha^{2n}-1}{\alpha^2-1}=\alpha^{2(n-1)}+\alpha^{2(n-2)}+\ldots+\alpha^2+1\geqslant$$
$$\geqslant n\sqrt[n]{\alpha^2\cdot\alpha^4\ldots\alpha^{2n-2}}$$

(using the theorem on the arithmetic and the geometric mean of several numbers).

Since

$$2+4+\ldots+(2n-2)=n\,(n-1),$$

we have indeed

$$\frac{\alpha^{2n}-1}{\alpha^2-1}\geqslant n\alpha^{n-1}.$$

69. Rewrite the sum in the following way

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{2^2}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{2^3}\right)+\ldots+$$
$$+\left(\frac{1}{2^{n-2}+1}+\ldots+\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n-1}\,.$$

Each of the bracketed expressions exceeds $\frac{1}{2}$ and, consequently, the total sum is more than $\frac{n}{2}$. On the other hand, the sum may be rewritten as

$$1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\ldots+$$
$$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n-1}\right).$$

But each of the bracketed expressions is less than unity, and, consequently, the total sum is less than n.

70. On transformation we get the inequality

$$(a+c)(a+b)(b+d)(c+d)-$$
$$-(a+b+c+d)(c+d)ab-$$
$$-(a+b+c+d)cd(a+b)\geqslant 0,$$

or the following one

$$(ad-bc)^2\geqslant 0.$$

SOLUTIONS TO SECTION 9

1. Putting in the basic formula $n=1$, we find

$$v_2=3v_1-2v_0=3\cdot3-2\cdot2=5=2^2+1.$$

Suppose that

$$v_k=2^k+1\quad(k=1, 2, \ldots, n),$$

and let us prove that

$$v_{n+1}=2^{n+1}+1.$$

Indeed

$$v_{n+1}=3v_n-2v_{n-1}=3(2^n+1)-2(2^{n-1}+1)=$$
$$=3\cdot2^n+3-2^n-2=2^n(3-1)+1=2^{n+1}+1.$$

2. Solved as the preceding problem.

3. As is easily seen, the required relation is indeed valid at $n=1$.

Assuming its validity at the subscript equal to n, let us prove that it is also valid at the subscript equal to $n+1$.

Indeed

$$\frac{a_{n+1}-\sqrt{A}}{a_{n+1}+\sqrt{A}}=\frac{\frac{1}{2}\left(a_n+\frac{A}{a_n}\right)-\sqrt{A}}{\frac{1}{2}\left(a_n+\frac{A}{a_n}\right)+\sqrt{A}}=$$

$$=\frac{a_n^2-2\sqrt{A}\,a_n+A}{a_n^2+2\sqrt{A}\,a_n+A}=\left(\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}\right)^2.$$

But by supposition

$$\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{n-1}}.$$

Therefore

$$\frac{a_{n+1}-\sqrt{A}}{a_{n+1}+\sqrt{A}}=\left(\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}\right)^2=$$

$$=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2\cdot 2^{n-1}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^n}.$$

4. We have

$$a_2=\frac{a_0+a_1}{2},\quad a_3=\frac{a_1+a_2}{2},\quad a_4=\frac{a_2+a_3}{2},\quad a_5=\frac{a_3+a_4}{2},\ \ldots .$$

Hence

$$a_2-a_1=\frac{a_0-a_1}{2},\quad a_3-a_2=\frac{a_1-a_2}{2},\quad a_4-a_3=\frac{a_2-a_3}{2},\ \ldots .$$

Consequently

$$a_2-a_1=-\frac{a_1-a_0}{2},$$

$$a_3-a_2=\frac{a_1-a_2}{2}=\frac{a_1-a_0}{2^2},$$

$$a_4-a_3=-\frac{a_1-a_0}{2^3},$$

$$\cdots\cdots\cdots\cdots$$

It is easy to see that there exists the following general formula

$$a_n-a_{n-1}=(-1)^{n-1}\frac{a_1-a_0}{2^{n-1}}.$$

Adding term by term all the last formulas, we have

$$a_n - a_1 = -\frac{a_1-a_0}{2} + \frac{a_1-a_0}{2^2} - \frac{a_1-a_0}{2^3} + \ldots + (-1)^{n-1}\frac{a_1-a_0}{2^{n-1}} =$$

$$= -\frac{a_1-a_0}{2}\left(1 - \frac{1}{2} + \frac{1}{2^2} + \ldots + (-1)^{n-2}\frac{1}{2^{n-2}}\right) =$$

$$= \frac{a_1-a_0}{3}\left\{(-1)^{n-1}\frac{1}{2^{n-1}} - 1\right\}.$$

Hence, finally,

$$a_n = \frac{2a_1+a_0}{3} + (-1)^{n-1}\frac{a_1-a_0}{3\cdot 2^{n-1}}.$$

5. Consider the relationship

$$a_k = 3a_{k-1} + 1.$$

Putting here k equal to 2, 3, 4, ..., n, we get

$$\sum_{k=2}^{n} a_k = 3\sum_{k=2}^{n} a_{k-1} + n - 1.$$

Put

$$a_1 + a_2 + \ldots + a_n = S.$$

We then have

$$S - a_1 = 3(S - a_n) + n - 1.$$

Consequently

$$S = \frac{1}{2}\{3a_n - a_1 - n + 1\}.$$

It remains to express a_n in terms of a_1. We have

$$a_n = 3a_{n-1} + 1, \quad a_{n-1} = 3a_{n-2} + 1.$$

Hence

$$a_n - a_{n-1} = 3(a_{n-1} - a_{n-2}).$$

Therefore

$$a_n - a_{n-1} = 3(a_{n-1} - a_{n-2}) = 3^2(a_{n-2} - a_{n-3}) =$$

$$= 3^3(a_{n-3} - a_{n-4}) = \ldots = 3^{n-2}(a_2 - a_1)$$

But

$$a_2 = 3a_1 + 1 = 7.$$

And so

$$a_n - a_{n-1} = 5 \cdot 3^{n-2}.$$

Putting here n equal to 2, 3, 4, . . ., n, we have

$$a_2 - a_1 = 5 \cdot 1,$$
$$a_3 - a_2 = 5 \cdot 3,$$
$$a_4 - a_3 = 5 \cdot 3^2,$$
$$\dots\dots\dots$$
$$a_n - a_{n-1} = 5 \cdot 3^{n-2}.$$

Adding these equalities termwise, we find

$$a_n - a_1 = 5\,(1 + 3 + 3^2 + \dots + 3^{n-2}) = \frac{5}{2}\,(3^{n-1} - 1).$$

Rewrite the expression for S in the following way

$$S = \frac{1}{2}\{3\,(a_n - a_1) + 2a_1 - n + 1\} = \frac{1}{2}\left\{\frac{15}{2}\,(3^{n-1} - 1) + 4 - n + 1\right\} = \frac{1}{4}\{5\,(3^n - 1) - 2n\}.$$

6. We have

$$a_n = ka_{n-1} + l,$$
$$a_{n-1} = ka_{n-2} + l.$$

Consequently

$$a_n - a_{n-1} = k\,(a_{n-1} - a_{n-2}) = k^2\,(a_{n-2} - a_{n-3}) = \dots = k^{n-2}\,(a_2 - a_1).$$

Hence

$$a_2 - a_1 = (a_2 - a_1),$$
$$a_3 - a_2 = k\,(a_2 - a_1),$$
$$a_4 - a_3 = k^2\,(a_2 - a_1),$$
$$\dots\dots\dots\dots$$
$$a_n - a_{n-1} = k^{n-2}\,(a_2 - a_1).$$

Adding these equalities, we find

$$a_n = k^{n-1}a_1 + \frac{k^{n-1} - 1}{k - 1}\,l.$$

7. Rewrite the given relationship in the following manner

$$a_{n+1} - a_n - (a_n - a_{n-1}) = 1.$$

Put

$$a_n - a_{n-1} = x_n \quad (n = 2, 3, 4, \ldots).$$

We then have

$$x_{n+1} - x_n = 1.$$

Putting here n equal to 2, 3, . . ., $n - 1$ and adding, we find

$$x_n - x_2 = n - 2.$$

Putting then in the equality

$$a_n - a_{n-1} = x_n$$

$n = 3, 4, \ldots, n$ and adding, we get

$$a_n - a_2 = x_3 + x_4 + \ldots + x_n.$$

And so

$$a_n = a_2 + x_3 + x_4 + \ldots + x_n.$$

But

$$\sum_{k=3}^{n} x_k = \sum_{k=3}^{n} (x_2 + k - 2) = (n-2)x_2 + (n-2) + \\ + (n-3) + \ldots + 1 = (n-2)x_2 + \frac{(n-1)(n-2)}{2}.$$

Hence

$$a_n = a_2 + (n-2)x_2 + \frac{(n-1)(n-2)}{2} = \\ = a_2 + (n-2)(a_2 - a_1) + \frac{(n-1)(n-2)}{2} = \\ = \frac{(n-1)(n-2)}{2} + (n-1)a_2 - (n-2)a_1.$$

8. Put

$$a_{n+2} - a_{n+1} = x_n.$$

Then the following relationship will take place

$$x_{n+1} - 2x_n + x_{n-1} = 1.$$

Using the result of the preceding problem we have

$$x_n = \frac{(n-1)(n-2)}{2} + (n-1)x_2 - (n-2)x_1.$$

But it is obvious that

$$a_n - a_2 = x_1 + x_2 + \ldots + x_{n-2} = \sum_{k=1}^{n-2} x_k.$$

Consequently

$$a_n - a_2 = \frac{1}{2}\sum_{k=1}^{n-2}(k-1)(k-2) + x_2\sum_{k=1}^{n-2}(k-1) - x_1\sum_{k=1}^{n-2}(k-2).$$

Finally

$$a_n = \frac{(n-1)(n-2)}{2}a_3 - (n-3)(n-1)a_2 + \frac{(n-2)(n-3)}{2}a_1 + \frac{(n-1)(n-2)(n-3)}{6}.$$

9. The required formulas can be deduced by the method of mathematical induction. It is evident that they take place at $n = 1$. Since

$$a_n = \frac{a_{n-1} + b_{n-1}}{2},$$

assuming that the formulas are valid at $n-1$, let us prove their validity at n. By supposition, we have

$$a_{n-1} = a + \frac{2}{3}(b-a)\left(1 - \frac{1}{4^{n-1}}\right),$$

$$b_{n-1} = a + \frac{2}{3}(b-a)\left(1 + \frac{1}{2\cdot 4^{n-1}}\right).$$

Then

$$a_n = \frac{a_{n-1} + b_{n-1}}{4} = a + \frac{2}{3}(b-a)\left(1 - \frac{1}{4^n}\right)$$

and, consequently, this formula takes place for any whole positive n. It only remains to prove that the formula for b_n is true for any whole positive n as well.

We have

$$b_n = \frac{a_n + b_{n-1}}{2} = a + \frac{2}{3}(b-a)\left(1 + \frac{1}{2\cdot 4^n}\right)$$

and the proof is completed.

However, this problem can be solved in quite a different way. It is obvious that

$$a_n = \frac{a_{n-1}+b_{n-1}}{2}, \quad b_n = \frac{a_{n-1}+3b_{n-1}}{4}.$$

Multiplying both members of these equalities by some factor λ, we get

$$a_n + \lambda b_n = \left(\frac{1}{2} + \frac{1}{4}\lambda\right) a_{n-1} + \left(\frac{1}{2} + \frac{3}{4}\lambda\right) b_{n-1}.$$

Let us choose λ so that

$$\frac{1}{2} + \frac{3}{4}\lambda = \left(\frac{1}{2} + \frac{1}{4}\lambda\right)\lambda.$$

There will be two required values of λ, and they will be the roots of the equation

$$\lambda^2 - \lambda - 2 = 0,$$

i.e. will be equal to $\lambda_1 = 2$ and $\lambda_2 = -1$.

And so, at these values of λ there exists the equality

$$a_n + \lambda b_n = \left(\frac{1}{2} + \frac{1}{4}\lambda\right)(a_{n-1} + \lambda b_{n-1}),$$

which holds true for all whole positive values of n. Putting here n consecutively equal to 1, 2, 3, ..., n, we get

$$a_1 + \lambda b_1 = \left(\frac{1}{2} + \frac{1}{4}\lambda\right)(a + \lambda b),$$

$$a_2 + \lambda b_2 = \left(\frac{1}{2} + \frac{1}{4}\lambda\right)(a_1 + \lambda b_1),$$

. . . .

$$a_n + \lambda b_n = \left(\frac{1}{2} + \frac{1}{4}\lambda\right)(a_{n-1} + \lambda b_{n-1}).$$

Multiplying these equalities termwise, we find

$$a_n + \lambda b_n = \left(\frac{1}{2} + \frac{1}{4}\lambda\right)^n (a + \lambda b),$$

for any whole positive n and at $\lambda=2$ and -1. Substituting these values of λ, we find

$$a_n+2b_n=a+2b,$$
$$a_n-b_n=\frac{1}{4^n}(a-b),$$

wherefrom we have indeed

$$a_n=a+\frac{2}{3}(b-a)\left(1-\frac{1}{4^n}\right),$$
$$b_n=a+\frac{2}{3}(b-a)\left(1+\frac{1}{2\cdot 4^n}\right).$$

10. We have

$$x_n=x_{n-1}+2\sin^2\alpha\, y_{n-1},$$
$$y_n=2\cos^2\alpha\, x_{n-1}+y_{n-1}.$$

Multiplying the second equality by λ and adding the first one, we get

$$x_n+\lambda y_n=(1+2\lambda\cos^2\alpha)\,x_{n-1}+(2\sin^2\alpha+\lambda)\,y_{n-1}.$$

Let us choose λ so that the following equality takes place

$$(2\sin^2\alpha+\lambda)=\lambda(1+2\lambda\cos^2\alpha).$$

Hence

$$\lambda=\pm\tan\alpha.$$

We then obtain

$$(x_n+\lambda y_n)=(1+2\lambda\cos^2\alpha)(x_{n-1}+\lambda y_{n-1})$$

or

$$(x_n+\lambda y_n)=(1+2\lambda\cos^2\alpha)^n(x_0+\lambda y_0).$$

Substituting the values of x_0 and y_0 and putting in succession $\lambda=\tan\alpha$ and $\lambda=-\tan\alpha$, we find the following two equalities

$$x_n+y_n\cdot\tan\alpha=(1+\sin 2\alpha)^n\sin\alpha,$$
$$x_n-y_n\cdot\tan\alpha=-(1-\sin 2\alpha)^n\sin\alpha.$$

Hence

$$x_n=\frac{1}{2}\sin\alpha\,\{(1+\sin 2\alpha)^n-(1-\sin 2\alpha)^n\},$$
$$y_n=\frac{1}{2}\cos\alpha\,\{(1+\sin 2\alpha)^n+(1-\sin 2\alpha)^n\}.$$

11. As in the two previous problems, we get

$$x_n + \lambda_1 y_n = \mu_1^n (x_0 + \lambda_1 y_0),$$
$$x_n + \lambda_2 y_n = \mu_2^n (x_0 + \lambda_2 y_0),$$

where $\mu_1 = \alpha + \lambda_1\gamma$, $\mu_2 = \alpha + \lambda_2\gamma$, λ_1 and λ_2 being the roots of the quadratic equation

$$(\beta + \lambda\delta) = \lambda (\alpha + \lambda\gamma).$$

If $\lambda_1 \neq \lambda_2$, then we have two equations for determining two unknowns x_n and y_n, and the problem is solved.

Let us now assume that $\lambda_1 = \lambda_2$. Then $\mu_1 = \mu_2$ and the two equations coincide. To determine x_n and y_n proceed as follows.

We have

$$x_n = -\lambda_1 y_n + \mu_1^n (x_0 + \lambda_1 y_0). \qquad (*)$$

Substituting the value of x_n into the second of the original equalities, we find

$$y_n = \gamma [-\lambda_1 y_{n-1} + \mu_1^{n-1} (x_0 + \lambda_1 y_0)] + \delta y_{n-1}.$$

Hence

$$y_n + (\gamma\lambda_1 - \delta)\, y_{n-1} = \gamma\mu_1^{n-1} (x_0 + \lambda_1 y_0).$$

Put $y_n = \mu_1^n z_n$. Then for z_n we obtain the following relation

$$\mu_1 z_n + (\gamma\lambda_1 - \delta)\, z_{n-1} = \gamma (x_0 + \lambda_1 y_0)$$

or

$$z_n = \frac{\delta - \gamma\lambda_1}{\mu_1} z_{n-1} + \frac{\gamma}{\mu_1} (x_0 + \lambda_1 y_0),$$

wherefrom we find z_n (see Problem 6) and then y_n; x_n is found by the formula (*).

12. Rewrite the given relationship in the following way

$$x_n - \alpha x_{n-1} - \beta x_{n-2} = 0.$$

Put

$$\alpha = a + b, \quad \beta = -ab$$

(i.e. a and b are the roots of the quadratic equation s^2 —

$-\alpha s-\beta=0$). Then we have

$$x_n - ax_{n-1} - bx_{n-1} + abx_{n-2} = 0,$$
$$x_n - ax_{n-1} - b(x_{n-1} - ax_{n-2}) = 0.$$

Put

$$x_n - ax_{n-1} = y_n.$$

The given relationship takes the form

$$y_n - by_{n-1} = 0.$$

Hence

$$y_n = by_{n-1},$$
$$y_{n-1} = by_{n-2},$$
$$\cdots\cdots\cdots$$
$$y_2 = by_1.$$

Consequently

$$y_n = b^{n-1}y_1.$$

For finding x_n we now have

$$x_n - ax_{n-1} = b^{n-1}y_1.$$

Put $x_n = b^n z_n$, then

$$bz_n - az_{n-1} = y_1$$

or

$$z_n = \frac{a}{b} z_{n-1} + \frac{y_1}{b}.$$

Using the result of Problem 6, we find

$$z_n = \left(\frac{a}{b}\right)^{n-1} z_1 + \frac{\left(\frac{a}{b}\right)^{n-1} - 1}{\frac{a}{b} - 1} \frac{y_1}{b}.$$

Performing simple transformations, we finally obtain

$$x_n = \frac{a^n - b^n}{a-b} x_1 - ab\,\frac{a^{n-1} - b^{n-1}}{a-b} x_0.$$

However, this problem can be solved by the method used in the previous problem, if we consider two sequences x_n and y_n defined by the relationships

$$x_n = \alpha x_{n-1} + \beta y_{n-1}, \quad y_n = 1 \cdot x_{n-1} + 0 \cdot y_{n-1}.$$

13. Solved as the preceding problem. In this case

$$a=1, \quad b=-\frac{q}{p+q}.$$

14. Considering the two variables y_n and z_n, determined by the relationships

$$y_n=\alpha y_{n-1}+\beta z_{n-1}, \quad z_n=\gamma y_{n-1}+\delta z_{n-1},$$

we put

$$\frac{y_n}{z_n}=x_n.$$

Then the variable x_n will satisfy the given relationship

$$x_n=\frac{\alpha x_{n-1}+\beta}{\gamma x_{n-1}+\delta},$$

and the solution of our problem will be reduced to that of Problem 11. For instance, in the given particular case

$$x_n=\frac{x_{n-1}+1}{x_{n-1}+3}$$

we have

$$y_n=y_{n-1}+z_{n-1}, \quad z_n=y_{n-1}+3z_{n-1}$$

and so on.

The second particular case

$$x_n=\frac{x_{n-1}}{2x_{n-1}+1}$$

is readily considered in the following way.

Rewrite this relationship as follows

$$\frac{1}{x_n}=\frac{2x_{n-1}+1}{x_{n-1}}=2+\frac{1}{x_{n-1}}.$$

Then

$$\frac{1}{x_n}-\frac{1}{x_{n-1}}=2.$$

Putting here $n=1, 2, 3, \ldots, n$ and adding, we get

$$\frac{1}{x_n}-\frac{1}{x_0}=2n, \quad x_n=\frac{x_0}{2nx_0+1}.$$

15. It is easily seen that

$$a_{n+1}b_{n+1}=a_nb_n,$$

and, consequently,

$$a_n b_n = a_0 b_0$$

at any whole n.

But

$$\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{a_n}+\sqrt{b_n}}=\frac{a_n-\sqrt{a_n b_n}}{a_n+\sqrt{a_n b_n}}=\frac{a_n-\sqrt{a_{n-1}b_{n-1}}}{a_n+\sqrt{a_{n-1}b_{n-1}}}=$$

$$=\frac{\frac{a_{n-1}+b_{n-1}}{2}-\sqrt{a_{n-1}b_{n-1}}}{\frac{a_{n-1}+b_{n-1}}{2}+\sqrt{a_{n-1}b_{n-1}}}=\left(\frac{\sqrt{a_{n-1}}-\sqrt{b_{n-1}}}{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}\right)^2.$$

Put $\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{a_n}+\sqrt{b_n}}=u_n$. Then we have

$$u_{n-1}=u_{n-2}^2,$$

$$u_{n-2}=u_{n-3}^2,$$

$$\cdots\cdots\cdots$$

$$u^2=u_1^2,$$

$$u_1=u_0^2.$$

Raising consecutively these equalities to the powers 1, 2, $2^2, \ldots, 2^{n-2}$, we find

$$u_{n-1}=u_0^{2^{n-1}}.$$

But

$$u_{n-1}=\frac{\sqrt{a_{n-1}}-\sqrt{b_{n-1}}}{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}=\frac{a_{n-1}-\sqrt{a_0 b_0}}{a_{n-1}+\sqrt{a_0 b_0}},$$

$$u_0=\frac{\sqrt{a_0}-\sqrt{b_0}}{\sqrt{a_0}+\sqrt{b_0}}=\frac{a_0-\sqrt{a_0 b_0}}{a_0+\sqrt{a_0 b_0}}.$$

Therefore we have

$$\frac{a_{n-1}-\sqrt{a_0 b_0}}{a_{n-1}+\sqrt{a_0 b_0}}=\left(\frac{a_0-\sqrt{a_0 b_0}}{a_0+\sqrt{a_0 b_0}}\right)^{2^{n-1}}.$$

16. We have

$$\frac{1}{(2k)^3-2k}=\frac{1}{2k}\cdot\frac{1}{(2k)^2-1}=\frac{1}{4k}\left[\frac{1}{2k-1}-\frac{1}{2k+1}\right]=$$
$$=\frac{1}{2}\left\{\frac{2k-(2k-1)}{2k(2k-1)}-\frac{(2k+1)-2k}{2k(2k+1)}\right\}=$$
$$=\frac{1}{2}\left\{\frac{1}{2k-1}-\frac{1}{2k}-\frac{1}{2k}+\frac{1}{2k+1}\right\}.$$

Therefore

$$\sum_{k=1}^{n}\frac{1}{(2k)^3-2k}=\frac{1}{2}\left\{\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)+\right.$$
$$\left.+\left(\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right)+\frac{1}{2n+1}-2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\right\}=$$
$$=\frac{1}{2}\left\{2\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)-1+\frac{1}{2n+1}-\right.$$
$$\left.-2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\right\}=\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right)-$$
$$-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)-\frac{n}{2n+1}.$$

Hence

$$\sum_{k=1}^{n}\frac{1}{(2k)^3-2k}+\frac{n}{2n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-$$
$$-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$$

(see Problem 33, Sec. 1).

17. Let us denote our expression by $\varphi_n(x)$. We have $\varphi_1(x)=(1-x)+x=1$,

$$\varphi_2(x)=(1-x)(1-x^2)+x(1-x^2)+x^2=1,$$

wherefrom we can assume that $\varphi_n(x)=1$ for any n. It is easily seen that the following relation takes place

$$\varphi_{n+1}(x)=(1-x^{n+1})\varphi_n(x)+x^{n+1}.$$

Assuming $\varphi_n(x)=1$, from the last relation we obtain $\varphi_{n+1}(x)=1$. But since $\varphi_1(x)=1$, it follows that $\varphi_n(x)=1$ for any whole positive n.

18. Put

$$\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\ldots+\frac{x^{2^{n-1}}}{1-x^{2^n}}=\varphi_n(x).$$

Then

$$\varphi_{n+1}(x)=\varphi_n(x)+\frac{x^{2^n}}{1-x^{2^{n+1}}}\cdot$$

Now it is easy to prove the required formula using the induction method.

19. Put

$$(1+x)(1+x^2)(1+x^{2^2})\ldots(1+x^{2^{n-1}})=X.$$

Multiplying both members by $1-x$, we find

$$X(1-x)=[(1-x)(1+x)](1+x^2)(1+x^{2^2})\ldots(1+x^{2^{n-1}})=$$
$$=[(1-x^2)(1+x^2)](1+x^{2^2})\ldots(1+x^{2^{n-1}})=$$
$$=[(1-x^4)(1+x^4)](1+x^8)\ldots(1+x^{2^{n-1}})=\ldots=1-x^{2^n}.$$

Hence

$$X=\frac{1-x^{2^n}}{1-x}=1+x+x^2+x^3+\ldots+x^{2^n-1}.$$

20. We have

$$1+\frac{1}{a}=\frac{a+1}{a},$$

$$1+\frac{1}{a}+\frac{a+1}{ab}=\frac{a+1}{a}+\frac{a+1}{ab}=\frac{(a+1)(b+1)}{ab}.$$

Let us assume that

$$1+\frac{1}{a}+\frac{a+1}{ab}+\ldots+\frac{(a+1)(b+1)\ldots(s+1)}{abc\ldots sk}=$$
$$=\frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots sk}.$$

Adding $\frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots skl}$ to both members, we get

$$\frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots sk}+\frac{(a+1)(b+1)\ldots(s+1)(k+1)}{abc\ldots skl}=$$

$$=\frac{(a+1)(b+1)\ldots(k+1)(l+1)}{abc\ldots skl},$$

and the formula is proved by the induction method.

21. We have

$$\frac{b}{a(a+b)}=\frac{(a+b)-a}{a(a+b)}=\frac{1}{a}-\frac{1}{a+b},$$

$$\frac{c}{(a+b)(a+b+c)}=\frac{(a+b+c)-(a+b)}{(a+b)(a+b+c)}=\frac{1}{a+b}-\frac{1}{a+b+c},$$

. . . .

$$\frac{l}{(a+b+\ldots+k)(a+b+\ldots+k+l)}=\frac{1}{a+b+\ldots+k}-$$

$$-\frac{1}{a+b+\ldots+k+l}.$$

Adding these equalities term by term, we find

$$\frac{b}{a(a+b)}+\frac{c}{(a+b)(a+b+c)}+\ldots+$$

$$+\frac{l}{(a+b+\ldots+k)(a+b+\ldots+k+l)}=$$

$$=\frac{1}{a}-\frac{1}{a+b+\ldots+k+l}=\frac{b+c+\ldots+k+l}{a(a+b+c+\ldots+k+l)}$$

and the identity is proved.

22. We have

$$F_1(z)=\frac{q}{1-q}(1-z),$$

$$F_1(qz)=\frac{q}{1-q}(1-qz).$$

Hence

$$1+F_1(z)-F_1(qz)=1+\frac{q}{1-q}(1-z)-\frac{q}{1-q}(1-qz)=1-qz,$$

i.e. the identity is true at $n=1$.

But

$$F_n(z)=F_{n-1}(z)+\frac{q^n}{1-q^n}(1-z)(1-qz)\ldots(1-q^{n-1}z),$$

$$F_n(qz)=F_{n-1}(qz)+\frac{q^n}{1-q^n}(1-qz)(1-q^2z)\ldots(1-q^nz).$$

Let us assume that the identity is true at $n-1$, i.e. that there exists the following equality

$$1+F_{n-1}(z)-F_{n-1}(qz)=(1-qz)(1-q^2z)\dots(1-q^{n-1}z).$$

We then have

$$1+F_n(z)-F_n(qz)=(1-qz)(1-q^2z)\dots(1-q^{n-1}z)+$$
$$+\frac{q^n}{1-q^n}(1-z)(1-qz)\dots(1-q^{n-1}z)-$$
$$-\frac{q^n}{1-q^n}(1-qz)(1-q^2z)\dots(1-q^nz)=$$
$$=(1-qz)(1-q^2z)\dots(1-q^{n-1}z)\left\{1+\frac{q^n}{1-q^n}(1-z)-\right.$$
$$\left.-\frac{q^n}{1-q^n}(1-q^nz)\right\}=(1-qz)(1-q^2z)\dots(1-q^{n-1}z)(1-q^nz),$$

which proves the identity for any n.

23. Put, as in the preceding problem,

$$F_n(z)=\frac{q}{1-q}(1-z)+\frac{q^2}{1-q^2}(1-z)(1-qz)+\dots+$$
$$+\frac{q^n}{1-q^n}(1-z)(1-qz)\dots(1-q^{n-1}z).$$

Hence

$$F_n(q^{-n})=\sum_{k=1}^{n}\frac{q^k}{1-q^k}\left(1-\frac{1}{q^n}\right)\left(1-\frac{q}{q^n}\right)\dots\left(1-\frac{q^{k-1}}{q^n}\right).$$

Let us prove that

$$F_n(q^{-n})=-n.$$

We have (see the identity of the preceding problem)

$$1+F_n(q^{-1})-F_n(1)=0.$$

But

$$F_n(1)=0.$$

Consequently

$$F_n(q^{-1})=-1.$$

Suppose

$$F_n(q^{-n+1})=-(n-1).$$

We have

$$1+F_n(q^{-n})-F_n(q^{-n+1})=0.$$

Hence

$$F_n(q^{-n})=F_n(q^{-n+1})-1=-(n-1)-1=-n.$$

And so indeed

$$\sum_{k=1}^{n}\frac{q^k}{1-q^k}\left(1-\frac{1}{q^n}\right)\left(1-\frac{q}{q^n}\right)\left(1-\frac{q^{k-1}}{q^n}\right)=-n.$$

Putting here $q^{-1}=a$, we get the required identity.

24. Put

$$u_k=\frac{a(a-1)\ldots(a-k+1)}{b(b-1)\ldots(b-k+1)},$$

$$u_{k+1}=\frac{a(a-1)\ldots(a-k+1)(a-k)}{b(b-1)\ldots(b-k+1)(b-k)}.$$

Hence

$$\frac{u_{k+1}}{u_k}=\frac{a-k}{b-k},\quad (b-k)u_{k+1}=(a-k)u_k.$$

Consequently

$$\sum_{k=1}^{n}u_k(a-k)=\sum_{k=1}^{n}u_{k+1}(b+1-k-1).$$

But

$$\sum_{k=1}^{n}u_k=S_n.$$

Therefore

$$aS_n-\sum_{k=1}^{n}ku_k=(b+1)\sum_{k=1}^{n}u_{k+1}-\sum_{k=1}^{n}(k+1)u_{k+1},$$

$$aS_n-\sum_{k=1}^{n}ku_k=(b+1)(S_n+u_{n+1}-u_1)-\sum_{k=2}^{n+1}ku_k.$$

Hence

$$(a-b-1)S_n=(b+1)(u_{n+1}-u_1)+$$
$$+u_1-(n+1)u_{n+1}=(b-n)u_{n+1}-bu_1.$$

Now S_n is readily found.

25. Proved easily by the induction method.

26. Both identities are easily proved by the induction method.

27. The left member is equal to

$$\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)=\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2}\cdot\frac{1}{2k-1}-\frac{1}{4k}\right)=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{2}\cdot\frac{1}{2k-1}-\frac{1}{2}\cdot\frac{1}{2k}\right)=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right).$$

28. If a sequence of numbers x_n is determined by the relationship

$$x_n=\alpha x_{n-1}+\beta x_{n-2}$$

at the given initial values x_0 and x_1, then there exists the following general expression for x_n

$$x_n=\frac{a^n-b^n}{a-b}x_1-ab\,\frac{a^{n-1}-b^{n-1}}{a-b}x_0,$$

where a and b are the roots of the quadratic equation

$$s^2-\alpha s-\beta=0$$

(see Problem 12).

In our case we have the following relationship

$$u_n=u_{n-1}+u_{n-2},$$

i.e. $\alpha=\beta=1$ and $u_0=0$, $u_1=1$. Therefore

$$u_n=\frac{a^n-b^n}{a-b},$$

where a and b are the roots of the equation $s^2-s-1=0$, so that we may put

$$a=\frac{1+\sqrt{5}}{2},\quad b=\frac{1-\sqrt{5}}{2}.$$

Finally,

$$u_n=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

Using this expression for u_n, we can easily check the validity of all the proposed relations (see Problem 6, Sec. 3). However the last expression for u_n can be obtained in a different way.

We shall consider the quantities $u_0, u_1, u_2, u_3, \ldots$ as coefficients of some infinite series

$$\varphi(x) = u_1 + u_2 x + u_3 x^2 + u_4 x^3 + \ldots + u_{n-1} x^{n-2} + u_n x^{n-1} + \ldots$$

or

$$\varphi(x) = \sum_{k=0}^{\infty} u_{k+1} x^k.$$

Further

$$x\varphi(x) = \sum_{k=0}^{\infty} u_{k+1} x^{k+1} = \sum_{k=1}^{\infty} u_k x^k,$$

$$x^2\varphi(x) = \sum_{k=0}^{\infty} u_{k+1} x^{k+2} = \sum_{k=2}^{\infty} u_{k-1} x^k.$$

Therefore

$$\varphi(x) - x\varphi(x) - x^2\varphi(x) =$$

$$= \sum_{k=2}^{\infty} (u_{k+1} - u_k - u_{k-1}) x^k + u_1 + u_2 x - u_1 x = 1.$$

Hence (since $u_{k+1} - u_k - u_{k-1} = 0$)

$$\varphi(x)(1 - x - x^2) = 1$$

and

$$\varphi(x) = \frac{1}{1 - x - x^2}.$$

But the expression $\frac{1}{1-x-x^2}$ can be represented in the following form (expanded into partial fractions)

$$\frac{1}{1-x-x^2} = \frac{1}{\alpha - \beta}\left\{\frac{\alpha}{1+\alpha x} - \frac{\beta}{1+\beta x}\right\}, \qquad (*)$$

where

$$\alpha = \frac{\sqrt{5}-1}{2}, \quad \beta = -\frac{\sqrt{5}+1}{2}.$$

On the other hand,

$$\frac{1}{1+\alpha x}=1-\alpha x+\alpha^2x^2+\ldots,$$
$$\frac{1}{1+\beta x}=1-\beta x+\beta^2x^2+\ldots.$$

Substituting these expressions into the equality (*), we find

$$\frac{1}{1-x-x^2}=\sum_{k=0}^{\infty}\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}x^k.$$

Therefore, indeed

$$u_{k+1}=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}.$$

By the way, all the ten identities of the present problem can be proved using the method of mathematical induction as well. Let us prove, for example, identities 7° and 10°. At $n=1$ we have

$$u_2^2=u_1u_2,$$

which is really true.

Let us assume that

$$u_1u_2+u_2u_3+\ldots+u_{2n-3}u_{2n-2}=u_{2n-2}^2,$$

and prove that

$$u_1u_2+u_2u_3+\ldots+u_{2n-3}u_{2n-2}+u_{2n-2}u_{2n-1}+$$
$$+u_{2n-1}u_{2n}=u_{2n}^2.$$

Indeed, by assumption we have

$$(u_1u_2+\ldots+u_{2n-3}u_{2n-2})+u_{2n-2}u_{2n-1}+u_{2n-1}u_{2n}=$$
$$=u_{2n-2}^2+u_{2n-2}u_{2n-1}+u_{2n-1}u_{2n}=$$
$$=u_{2n-2}(u_{2n-2}+u_{2n-1})+u_{2n-1}u_{2n}=$$
$$=u_{2n-2}u_{2n}+u_{2n-1}u_{2n}=$$
$$=u_{2n}(u_{2n-2}+u_{2n-1})=u_{2n}^2.$$

Now, as far as identity 10° is concerned, it is readily checked at $n=1$.

Let us assume that

$$u_{n-1}^4 - u_{n-3}u_{n-2}u_nu_{n+1} = 1,$$

and prove that

$$u_n^4 - u_{n-2}u_{n-1}u_{n+1}u_{n+2} = 1.$$

To this end it is sufficient to prove that

$$u_n^4 - u_{n-1}^4 + u_{n-3}u_{n-2}u_nu_{n+1} - u_{n-2}u_{n-1}u_{n+1}u_{n+2} = 0.$$

But we have

$$\begin{aligned}u_n^4 - u_{n-1}^4 &+ u_{n-3}u_{n-2}u_nu_{n+1} - u_{n-2}u_{n-1}u_{n+1}u_{n+2} = \\ &= (u_n^2 + u_{n-1}^2)(u_n + u_{n-1})(u_n - u_{n-1}) + \\ &+ u_{n-2}u_{n+1}(u_{n-3}u_n - u_{n-1}u_{n+2}) = \\ &= u_{n+1}u_{n-2}\{u_n^2 + u_{n-1}^2 + u_{n-3}u_n - u_{n-1}u_{n+2}\} = \\ &= u_{n+1}u_{n-2}\{u_{n-1}^2 - u_{n-1}u_{n+2} + u_n(u_n + u_{n-3})\} = \\ &= u_{n+1}u_{n-2}\{u_{n-1}^2 - u_{n-1}u_{n+2} + 2u_nu_{n-1}\} = \\ &= u_{n+1}u_{n-2}u_{n-1}\{u_{n-1} - u_{n+2} + 2u_n\} = 0\end{aligned}$$

since

$$u_{n-1} - u_{n+2} + 2u_n = 0.$$

29. We have

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{2}{1\cdot 3} + \ldots + \frac{u_{n+2}}{u_{n+1}u_{n+3}} &= \sum_{k=0}^{n} \frac{u_{k+2}}{u_{k+1}u_{k+3}} = \\ &= \sum_{k=0}^{n} \frac{u_{k+3} - u_{k+1}}{u_{k+1}u_{k+3}} = \sum_{k=0}^{n} \left(\frac{1}{u_{k+1}} - \frac{1}{u_{k+3}}\right) = \\ &= \left(\frac{1}{u_1} + \frac{1}{u_2} + \ldots + \frac{1}{u_{n+1}}\right) - \left(\frac{1}{u_3} + \frac{1}{u_4} + \ldots + \frac{1}{u_{n+3}}\right) = \\ &= \frac{1}{u_1} + \frac{1}{u_2} - \frac{1}{u_{n+2}} - \frac{1}{u_{n+3}} = \frac{u_1 + u_2}{u_1u_2} - \frac{u_{n+2} + u_{n+3}}{u_{n+2}u_{n+3}} = \\ &= \frac{u_3}{u_1u_2} - \frac{u_{n+4}}{u_{n+2}u_{n+3}}\end{aligned}$$

30. Consider the sequence of numbers

$$v_0, \quad v_1, \quad v_2, \quad v_3, \quad v_4, \ldots,$$

determined by the following relationship

$$v_{n+1} = v_n + v_{n-1}.$$

We then have

$$\begin{aligned} v_2 &= v_0 + v_1, \\ v_3 &= v_2 + v_1 = v_0 + 2v_1, \\ v_4 &= v_3 + v_2 = 2v_0 + 3v_1, \\ v_5 &= v_4 + v_3 = 3v_0 + 5v_1, \\ &\dots\dots\dots\dots \end{aligned}$$

Using the method of induction, it is easy to get that in general

$$v_n = u_{n-1} \cdot v_0 + u_n v_1.$$

Consider the following sequence

$$v_0 = u_{p-1}, \quad v_1 = u_p, \quad \dots, v_n = u_{p+n-1}.$$

Then we have

$$v_n = u_{p+n-1} = u_{n-1}u_{p-1} + u_n u_p,$$

and formula 1° is proved.

Formula 2° follows from 1° at $p = n$. The proof of formula 3° is reduced to the proof of the following equality

$$u_n^2 + u_{n-1}^2 = u_n u_{n+1} - u_{n-2} u_{n-1}.$$

31. On the basis of formula 1° of the preceding problem we have

$$u_{3n} = u_{n-1} u_{2n} + u_n u_{2n+1}.$$

Thus, it is required to prove that

$$u_{n-1} \cdot u_{2n} + u_n \cdot u_{2n+1} = u_n^3 + u_{n+1}^3 - u_{n-1}^3.$$

The proof is rather simple if only the following relations are taken into account

$$\begin{aligned} u_{2n+1} &= u_{n+1}^2 + u_n^2, \\ u_{2n} &= u_{n-1} u_n + u_n u_{n+1}. \end{aligned}$$

32. Put

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} C_{n-k-1}^k = v_n.$$

We have to prove that $v_n = u_n$ (where u_n is the nth term of the Fibonacci series). Let us prove that for any n there will be

$$v_{n+1} = v_n + v_{n-1}.$$

Let us first assume that n is even and put $n = 2l$. We have

$$v_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} C^k_{n-k}, \quad v_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} C^k_{n-k-1}, \quad v_{n-1} = \sum_{k=0}^{\left[\frac{n-2}{2}\right]} C^k_{n-k-2}.$$

Since $n = 2l$,

$$\left[\frac{n}{2}\right] = l, \quad \left[\frac{n-1}{2}\right] = l-1, \quad \left[\frac{n-2}{2}\right] = l-1.$$

Therefore we have

$$v_n + v_{n-1} = \sum_{k=0}^{l-1} C^k_{n-k-1} + \sum_{k=0}^{l-1} C^k_{n-k-2}.$$

Put in the second sum $k = k' - 1$, then

$$v_n + v_{n-1} = 1 + \sum_{k=1}^{l-1} C^k_{n-k-1} + \sum_{k'=1}^{l} C^{k'-1}_{n-k'-1} =$$

$$= 1 + \sum_{k=1}^{l-1} (C^k_{n-k-1} + C^{k-1}_{n-k-1}) + C^{l-1}_{n-l-1}.$$

But, as is known,

$$C^k_{n-k-1} + C^{k-1}_{n-k-1} = C^k_{n-k}.$$

Therefore

$$v_n + v_{n-1} = 1 + \sum_{k=1}^{l-1} C^k_{n-k} + C^{l-1}_{l-1} = \sum_{k=0}^{l} C^k_{n-k} = v_{n+1},$$

since

$$C^{l-1}_{l-1} = 1 = C^l_l.$$

Likewise we prove that $v_{n+1} = v_n + v_{n-1}$ for odd n's as well. But it is easy to check that

$$v_1 = u_1, \quad v_2 = u_2.$$

Therefore it is obvious that

$$v_n = u_n$$

for any n.

33. Let us denote the number of whole positive solutions of our equation by $N_n(m)$. As is easily seen, $N_1(m) = 1$. Compute $N_2(m)$, i.e. the number of solutions to the equation

$$x_1 + x_2 = m.$$

In this equation x_1 can attain the following values: $1, 2, 3, \ldots, m-1$ and, consequently, the equation has the following system of solutions

$$(1, m-1), \quad (2, m-2), \ldots, \quad (m-1, 1),$$

i.e.

$$N_2(m) = m - 1.$$

Let us now pass over to computing $N_3(m)$, i.e. to determining the number of solutions of the equation

$$x_1 + x_2 + x_3 = m.$$

Let x_3 attain the values $1, 2, 3, \ldots, m-2$. It is clear that

$$N_3(m) = N_2(m-1) + N_2(m-2) + \ldots + N_2(2) =$$
$$= (m-2) + (m-3) + \ldots + 1 = \frac{(m-1)(m-2)}{1 \cdot 2} = C_{m-1}^2.$$

Using the induction method, we prove that

$$N_n(m) = C_{m-1}^{n-1} = \frac{(m-1)(m-2)\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \ldots (n-1)}.$$

It is obvious that

$$N_n(m) = N_{n-1}(m-1) + N_{n-1}(m-2) + \ldots + N_{n-1}(n-1).$$

Assuming that

$$N_{n-1}(m) = C_{m-1}^{n-2},$$

we have

$$N_n(m) = C_{m-2}^{n-2} + C_{m-3}^{n-2} + \ldots + C_{n-2}^{n-2} = C_{m-1}^{n-1}$$

(see Problem 70, Sec. 6).

34. The general form of the equations under consideration will be

$$kx + (k+1)y = n - k + 1 \quad (k = 1, 2, \ldots, n+1). \qquad (*)$$

Let us rewrite this equation as follows

$$k(x+y+1)+y=n+1$$

and put

$$x+y+1=z.$$

Then

$$y=n+1-kz,$$
$$x=(k+1)z-(n+2).$$

Whatever z may be these expressions yield solutions to the equation (*). Let us see what values must be attained by z for x and y to be whole and non-negative. And so, the following inequalities must take place

$$(n+1)-kz\geqslant 0,\quad (k+1)z-(n+2)\geqslant 0.$$

Hence

$$\frac{n+2}{k+1}\leqslant z\leqslant\frac{n+1}{k},$$

and z must be a whole number. If $n+2$ is not divisible by $k+1$, then z takes on the following values

$$\left[\frac{n+2}{k+1}\right]+1,\quad \left[\frac{n+2}{k+1}\right]+2,\quad \ldots,\quad \left[\frac{n+1}{k}\right].$$

Let us denote the number of solutions of the equation (*) by N_k. In this case we have

$$N_k=\left[\frac{n+1}{k}\right]-\left[\frac{n+2}{k+1}\right].$$

If $n+2$ is divisible exactly by $k+1$, then

$$N_k=\left[\frac{n+1}{k}\right]-\frac{n+2}{k+1}+1.$$

But if $n+2$ is not divisible by $k+1$, then

$$\left[\frac{n+2}{k+1}\right]=\left[\frac{n+1}{k+1}\right];$$

and if $n+2$ is divisible by $k+1$, then

$$\frac{n+2}{k+1}-1=\left[\frac{n+1}{k+1}\right].$$

Thus in all the cases

$$N_k=\left[\frac{n+1}{k}\right]-\left[\frac{n+1}{k+1}\right].$$

And so, the total number of solutions is equal to

$$N_1+N_2+\ldots+N_{n+1}=\left[\frac{n+1}{1}\right]-\left[\frac{n+1}{2}\right]+$$
$$+\left[\frac{n+1}{2}\right]-\left[\frac{n+1}{3}\right]+\ldots+\left[\frac{n+1}{n}\right]-\left[\frac{n+1}{n+1}\right]+$$
$$+\left[\frac{n+1}{n+1}\right]-\left[\frac{n+1}{n+2}\right]=\left[\frac{n+1}{1}\right]-\left[\frac{n+1}{n+2}\right]=n+1.$$

However, this result can be obtained in a different way. We have

$$\frac{1}{1-q^k}=\sum_{x=0}^{\infty} q^{kx}, \quad \frac{1}{1-q^{k+1}}=\sum_{y=0}^{\infty} q^{(k+1)y}.$$

Therefore

$$\frac{q^{k-1}}{(1-q^k)\,(1-q^{k+1})}=\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} q^{kx+(k+1)y+k-1}.$$

If we expand the right member of this equality in powers of q, then it is easily seen that the coefficient of q^n in this expansion will be equal to N_k, i.e. to the number of solutions of the equation

$$kx+(k+1)\,y=n-k+1.$$

Thus, the quantity

$$N_1+N_2+\ldots+N_{n+1}$$

will be the coefficient of q^n in the following expansion

$$\frac{1}{(1-q)(1-q^2)}+\frac{q}{(1-q^2)(1-q^3)}+\frac{q^2}{(1-q^3)(1-q^4)}+\ldots+$$
$$+\frac{q^n}{(1-q^{n+1})(1-q^{n+2})}+\frac{q^{n+1}}{(1-q^{n+2})(1-q^{n+3})}+\ldots.$$

But it is easily seen, that this expansion is equal to

$$\frac{1}{q\,(1-q)}\sum_{k=0}^{\infty}\left(\frac{1}{1-q^{k+1}}-\frac{1}{1-q^{k+2}}\right)=$$
$$=\frac{1}{q\,(1-q)}\left(\frac{1}{1-q}-1\right)=\sum_{n=0}^{\infty}(n+1)\,q^n.$$

Hence

$$N_1 + N_2 + \ldots + N_{n+1} = n + 1.$$

35. The general form of the equations will be

$$k^2 x + (k+1)^2 y = [(k+1)^2 - k^2]\, n - k^2$$
$$(k = 1, 2, 3, \ldots, n).$$

A direct substitution shows that one of the solutions will be

$$x = -(n+1), \quad y = n.$$

Then, as is known, all the solutions will be obtained from the expressions

$$x = -(n+1) + (p+1)^2 t, \quad y = n - p^2 t,$$

where p is one of the values attained by k.

For x and y to be non-negative it is necessary and sufficient that t attains whole values satisfying the inequalities

$$\frac{n+1}{(p+1)^2} \leqslant t \leqslant \frac{n}{p^2}.$$

Considering then separately two cases ($n + 1$ is divisible by $(p+1)^2$ and $n + 1$ is not divisible by $(p+1)^2$), we come to the desired result.

36. By hypothesis the black balls alternate with the white ones. Therefore, two suppositions are possible:

(1) the white balls occupy odd positions, i.e. the first, third, . . ., and the black balls even positions;

(2) the white balls occupy even positions, and the black balls odd positions.

It is easily seen that the white balls numbered $1, 2, \ldots, n$ can occupy odd positions in $n!$ ways, likewise the black balls can occupy even positions also in $n!$ ways. And so, according to the first assumption, we have $(n!)^2$ ways of arrangement of all the balls.

The second assumption yields the same number of arrangements. Hence, the total number of arrangements of the balls is $2\,(n!)^2$.

37. Let L_{nk}^{k} denote the number of ways in which kn distinct objects can be distributed into k groups of n objects in each group.

In how many ways is it possible to make up the first group of n objects? It is clear that the total number of the distinct combinations is equal to C^n_{nk}, and it is obvious that

$$L^k_{nk} = C^n_{nk} L^{k-1}_{nk-n}.$$

Hence

$$L^n_{nk} = C^n_{nk} C^n_{(k-1)n} \dots C^n_{2n}.$$

38. Let us consider the number of permutations of n elements in which two definite elements a and b are found side by side. The following cases are possible: (1) a occupies the first place, a occupies the second place, . . ., finally, a occupies $(n-1)$th place, and b is always on its right, i.e. in the second, third, . . ., nth place, respectively; (2) b occupies the first place, . . ., finally b occupies $(n-1)$th place, in all cases followed by a. Thus, the total number of cases amounts to $2(n-1)$, each case corresponding to $(n-2)!$ permutations. Therefore the total number of the permutations in which two definite elements a and b occur side by side will amount to

$$(n-2)!\ 2(n-1) = 2(n-1)!.$$

Consequently, the number of permutations of n elements in which two elements a and b are not found side by side will amount to

$$n! - 2(n-1)! = (n-1)!\ (n-2).$$

39. Let us denote the number of the required permutations by Q_n and put $n! = P_n$. Consider the whole totality of the permutations P_n. Among them there exist Q_n permutations in which none of the elements occupies its original position. Let us find the number of the permutations in which *only one element* retains its original position. Undoubtedly, this number will amount to nQ_{n-1}. Likewise, the number of permutations with only two definite elements retaining their original position will amount to $\frac{n(n-1)}{1\cdot 2} Q_{n-2}$, and so on. Finally, the number of permutations where all the elements retain the original position is $Q_0 = 1$. Thus, we have

$$P_n = Q_n + nQ_{n-1} + \frac{n(n-1)}{1\cdot 2} Q_{n-2} + \dots + nQ_1 + Q_0.$$

This equality can be written symbolically as

$$P^n = (Q+1)^n.$$

Here after involution all the exponents (superscripts) should be replaced by subscripts, so that Q^k turns into Q_k. Consequently, we can write the following symbolic identity valid for all values of x

$$(P+x)^n = (Q+1+x)^n$$

(since symbolically the power of P can be replaced everywhere by the same power of $Q+1$).

Putting here $x=-1$, we find

$$Q^n = (P-1)^n.$$

Passing over from the symbolic equality to an ordinary one, we have

$$Q_n = P_n - \frac{n}{1} P_{n-1} + \frac{n(n-1)}{1\cdot 2} P_{n-2} + \ldots + (-1)^{n-1} nP_1 + (-1)^n,$$

$$Q_n = n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right).$$

40. Consider all such permutations of n letters in which vacant squares may occur along with occupied ones. If $n=1$, then the number of ways in which one letter can be placed in r squares is equal to r (the first square is occupied by one letter, the rest of the squares being vacant; the second square is occupied by one letter, the rest of the squares being vacant, and so on). All permutations of two letters in r squares are obtained from just considered r permutations by placing the second letter in succession in the first, second, ..., rth square. Thus, the number of permutations of two letters in r squares will amount to r^2, and, as is easily seen, the total number of permutations of n letters in r squares will be equal to r^n. Let us denote by A_r the number of ways in which n distinct letters can be distributed in r squares so that each square contains at least one letter. The number of such permutations amounts to A_r. Then we shall consider all those permutations in which one and only one square is vacant. Their number is equal to rA_{r-1}. Further, the number of permutations where two and only

two squares are vacant is equal to

$$\frac{r(r-1)}{1\cdot 2} A_{r-2},$$

and so on.

Therefore we have

$$A_r + rA_{r-1} + \frac{r(r-1)}{1\cdot 2} A_{r-2} + \ldots + rA_1 + 1 = r^n + 1.$$

This equality can be written symbolically in the following way

$$(A+1)^r = r^n + 1$$

(i.e. after expanding the left member A^k should be throughout replaced by A_k).

Further, we have

$$(A+1+x)^r = \sum_{k=0}^{r} C_r^k x^k (A+1)^{r-k}.$$

This equality yields the following symbolic one which holds true for all values of x

$$(A+1+x)^r = \sum_{k=0}^{r} C_r^k x^k [(r-k)^n + 1].$$

Put here $x = -1$. Then

$$A^r = \sum_{k=0}^{r} C_r^k (-1)^k [(r-k)^n + 1] =$$

$$= \sum_{k=0}^{r} (-1)^k (r-k)^n C_r^k + \sum_{k=0}^{r} (-1)^k C_r^k.$$

But

$$\sum_{k=0}^{r} (-1)^k C_r^k = (1-1)^r = 0.$$

Therefore

$$A^r = \sum_{k=0}^{r} (-1)^k (r-k)^n C_r^k.$$

Passing over from the symbolic equality to an ordinary one, we get

$$A_r = \sum_{k=0}^{r} (-1)^k (r-k)^n C_r^k =$$

$$= r^n - \frac{r}{1}(r-1)^n + \frac{r(r-1)}{1\cdot 2}(r-2)^n - \ldots + (-1)^{r-1} r$$

(see Problem 55, Sec. 6).

SOLUTIONS TO SECTION 10

1. Put $a = \frac{1}{b}$, so that $|b| > 1$. Let us prove that

$$|b|^n > 1 + n(|b|-1) \quad (n > 1).$$

Indeed

$$|b|^n = \{1 + (|b|-1)\}^n = 1 + n(|b|-1) + \frac{n(n-1)}{1\cdot 2}(|b|-1)^2 + \ldots ,$$

wherefrom it follows that

$$|b|^n > 1 + n(|b|-1) \quad (n > 1).$$

Then

$$|x_n| = |a|^n = \frac{1}{|b|^n} < \frac{1}{1+n(|b|-1)}$$

and indeed

$$\lim_{n\to\infty} x_n = 0.$$

2. It is easily seen, that we may assume $a > 0$. Then $x_i > 0$ ($i = 1, 2, 3, \ldots$). Let k be a whole number satisfying the condition $k \leqslant a < k+1$, so that $\frac{a}{k+1} < 1$.

Put $n > k$. Then

$$\frac{a^n}{n!} = \frac{a^k}{1\cdot 2\cdot 3\ldots k}\cdot\frac{a}{k+1}\cdot\frac{a}{k+2}\cdots\frac{a}{n}.$$

But

$$\frac{a}{k+2} < \frac{a}{k+1}, \quad \frac{a}{k+3} < \frac{a}{k+1}, \quad \ldots, \quad \frac{a}{n} < \frac{a}{k+1}.$$

Therefore

$$\frac{a^n}{n!} < \frac{a^k}{k!}\left(\frac{a}{k+1}\right)^{n-k}.$$

But since $\frac{a}{k+1} < 1$, it follows that $\left(\frac{a}{k+1}\right)^{n-k} \to 0$, if $n \to \infty$, and therefore at any real a we have

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0,$$

i.e. the factorial $n!$ increases faster than the nth power of any real number.

3. Both the numerator and denominator of this fraction increase without bound along with an increase in n. Consider separately three cases: $k = h$, $k < h$ and $k > h$.

1° $k = h$. Divide the numerator and denominator by $n^k = n^h$. We get

$$\lim \frac{a_0 n^h + a_1 n^{h-1} + \dots + a_h}{b_0 n^h + b_1 n^{h-1} + \dots + b_h} = \lim \frac{a_0 + \frac{a_1}{n} + \dots + \frac{a_h}{n^h}}{b_0 + \frac{b_1}{n} + \dots + \frac{b_h}{n^h}} = \frac{a_0}{b_0}.$$

2° $k < h$.

$$\lim \frac{a_0 n^k + a_1 n^{k-1} + \dots + a_k}{b_0 n^h + b_1 n^{h-1} + \dots + b_h} = \lim \frac{\frac{a_0}{n^{h-k}} + \dots + \frac{a_k}{n^h}}{b_0 + \frac{b_1}{n} + \dots + \frac{b_h}{n^h}} = 0.$$

3° $k > h$. Analogously we get in this case

$$\frac{a_0 n^k + a_1 n^{k-1} + \dots + a_k}{b_0 n^h + b_1 n^{h-1} + \dots + b_h} \to \infty.$$

4. We have

$$\prod_{k=2}^{n} \frac{k^3-1}{k^3+1} = \prod_{k=2}^{n} \frac{k-1}{k+1} \cdot \prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1}.$$

But

$$\prod_{k=2}^{n} \frac{k-1}{k+1} = \frac{1\cdot 2\cdot 3\dots(n-1)}{3\cdot 4\cdot 5\dots(n+1)} = \frac{2}{n(n+1)},$$

$$\prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1} = \frac{7\cdot 13\cdot 21\dots(n^2+n+1)}{3\cdot 7\cdot 13\dots(n^2-n+1)} = \frac{n^2+n+1}{3}.$$

Therefore

$$\lim_{n\to\infty} P_n = \frac{2}{3}\lim \frac{n^2+n+1}{n^2+n} = \frac{2}{3}.$$

5. Put

$$\frac{1^k+2^k+3^k+\ldots+n^k}{n^{k+1}} = P_n^k.$$

At $k=1$ we have $P_n^1 = \frac{n+1}{2n}$ and consequently

$$\lim_{n\to\infty} P_n^1 = \frac{1}{2}.$$

Likewise we easily find $\lim\limits_{n\to\infty} P_n^2 = \frac{1}{3}$. Let us assume that $\lim\limits_{n\to\infty} P_n^i = \frac{1}{i+1}$ for all the values of i less than k, and prove that $\lim P_n^k = \frac{1}{k+1}$. Put $s_i = 1^i+2^i+\ldots+n^i$. We then have the following formula (see Problem 26, Sec. 7).

$$(k+1)\,s_k + \frac{(k+1)\,k}{1\cdot 2}\,s_{k-1} + \frac{(k+1)\,k\,(k-1)}{1\cdot 2\cdot 3}\,s_{k-2} + \ldots + \\ + (k+1)\,s_1 + s_0 = (n+1)^{k+1} - 1.$$

But $P_n^k = \frac{s_k}{n^{k+1}}$, therefore we have

$$P_n^k = \frac{1}{k+1}\left(1+\frac{1}{n}\right)^{k+1} - \frac{1}{(k+1)\,n^{k+1}} - \\ - \frac{k}{1\cdot 2}\,\frac{P_n^{k-1}}{n} - \ldots - \frac{1}{k+1}\,\frac{P_n^0}{n^k},$$

wherefrom it follows that

$$\lim_{n\to\infty} P_n^k = \frac{1}{k+1}.$$

This proposition can be proved directly. Let us make use of the inequality (see Problem 50, Sec. 8)

$$mx^{m-1}(x-1) > x^m - 1 > m(x-1)$$

($x>0$, not equal to 1, m is rational and does not lie between 0 and 1).

Put here $m=k+1$ and replace x by $\frac{x}{y}$. We get

$$(k+1)\,x^k(x-y) > x^{k+1} - y^{k+1} > (k+1)\,y^k(x-y).$$

Put here first $x=p$, $y=p-1$ and then $x=p+1$, $y=p$. We then find

$$(p+1)^{k+1}-p^{k+1}>(k+1)p^k>p^{k+1}-(p-1)^{k+1}.$$

Putting in this inequality $p=1, 2, \ldots, n$ and adding, we obtain

$$(n+1)^{k+1}-1>(k+1)(1^k+2^k+\ldots+n^k)>n^{k+1}.$$

Dividing all members of the inequality by $(k+1)n^{k+1}$, we find

$$\frac{1}{k+1}\left\{\left(1+\frac{1}{n}\right)^{k+1}-\frac{1}{n^{k+1}}\right\}>\frac{1^k+2^k+\ldots+n^k}{n^{k+1}}>\frac{1}{k+1}.$$

Hence it follows that

$$\lim_{n\to\infty}\frac{1^k+2^k+\ldots+n^k}{n^{k+1}}=\frac{1}{k+1}.$$

6. Using the notation of the preceding problem, we get

$$\frac{1^k+2^k+\ldots+n^k}{n^k}-\frac{n}{k+1}=n\left(P_n^k-\frac{1}{k+1}\right).$$

Making use of the expression for P_n^k obtained in the preceding problem, we have

$$n\left(P_n^k-\frac{1}{k+1}\right)=$$

$$=\frac{(n+1)^{k+1}-n^{k+1}}{(k+1)n^k}-\frac{1}{(k+1)n^k}-\frac{k}{2}P_n^{k-1}-\ldots-\frac{1}{k+1}\frac{P_n^0}{n^{k-1}}.$$

Hence

$$\lim n\left(P_n^k-\frac{1}{k+1}\right)=\lim\left\{\frac{(n+1)^{k+1}-n^{k+1}}{(k+1)n^k}-\frac{k}{2}P_n^{k-1}\right\}=\frac{1}{2},$$

since

$$\lim_{n\to\infty}\frac{(n+1)^{k+1}-n^{k+1}}{(k+1)n^k}=1 \text{ and } \lim_{n\to\infty}P_n^{k-1}=\frac{1}{k}.$$

7. From Problem 4, Sec. 9 we have

$$x_n=\frac{2x_1+x_0}{3}+(-1)^{n-1}\frac{(x_1-x_0)}{3\cdot 2^{n-1}},$$

wherefrom follows

$$\lim_{n\to\infty}x_n=\frac{x_0+2x_1}{3}.$$

8. We have the following relationship (see Problem 3, Sec. 9)

$$\frac{x_n-\sqrt{N}}{x_n+\sqrt{N}}=\left(\frac{x_0-\sqrt{N}}{x_0+\sqrt{N}}\right)^{2^n}.$$

Since $\left|\frac{x_0-\sqrt{N}}{x_0+\sqrt{N}}\right|<1$, we have

$$\lim_{n\to\infty}\left(\frac{x_0-\sqrt{N}}{x_0+\sqrt{N}}\right)^{2^n}=0.$$

Hence

$$\lim_{n\to\infty}\frac{x_n-\sqrt{N}}{x_n+\sqrt{N}}=0 \quad\text{and}\quad \lim_{n\to\infty}x_n=\sqrt{N}.$$

And so, we get a method for finding the square root of a number. It consists in the following: designate any positive number (say, the approximate value of a root accurate to unity) by x_0. We represent N in the form of a product of two factors, one of which is equal to x_0 so that

$$N=x_0\cdot\frac{N}{x_0}.$$

We take the arithmetic mean of these factors and denote it by x_1, so that

$$x_1=\frac{1}{2}\left(x_0+\frac{N}{x_0}\right).$$

Then we put

$$N=x_1\cdot\frac{N}{x_1},$$

and take the arithmetic mean once again

$$x_2=\frac{1}{2}\left(x_1+\frac{N}{x_1}\right)$$

and so on.

The error, which we introduce when taking x_n for an approximate value of $\sqrt{N}$, can be determined from the formula

$$\frac{x_n-\sqrt{N}}{x_n+\sqrt{N}}=\left(\frac{x_0-\sqrt{N}}{x_0+\sqrt{N}}\right)^{2^n}.$$

9. Let us first of all prove that

$$x_p^m > N.$$

Indeed

$$x_p^m = x_{p-1}^m \left(1 + \frac{N - x_{p-1}^m}{m x_{p-1}^m}\right)^m.$$

But

$$\left(1 + \frac{N - x_{p-1}^m}{m x_{p-1}^m}\right)^m > 1 + \frac{N - x_{p-1}^m}{x_{p-1}^m} = \frac{N}{x_{p-1}^m}$$

(see Problem 51, Sec. 8).

Therefore

$$x_p^m > N$$

for any whole positive p.

Let us now prove that x_p is a decreasing variable, i.e. prove that

$$x_p - x_{p-1} < 0.$$

Indeed

$$x_p - x_{p-1} = \frac{N - x_{p-1}^m}{m x_{p-1}^m} < 0.$$

And so, the variable x_n decreases but remains positive. Therefore it has a limit. Designate this limit by λ. From the relation

$$x_n = \frac{m-1}{m} x_{n-1} + \frac{N}{m x_{n-1}^{m-1}},$$

as $n \to \infty$, we get

$$\lambda = \frac{m-1}{m} \lambda + \frac{N}{m \lambda^{m-1}}, \quad \lambda^m = N \text{ and } \lambda = \sqrt[m]{N}.$$

It is obvious that

$$x_n > \sqrt[m]{N} > \frac{N}{x_n^{m-1}},$$

which enables us to find the upper limit of the error introduced as a result of taking x_n for an approximate value of $\sqrt[m]{N}$.

10. We have

$$0 < \sqrt[n]{\frac{1}{n!}} \leqslant \frac{1}{\sqrt{n}}$$

(see Problem 4, Sec. 8).

Hence follows the required result.

11. It is easy to prove the following inequality

$$\frac{x}{2+x} < \sqrt{1+x} - 1 < \frac{x}{2} \quad (1+x > 0).$$

Putting here $x = \frac{k}{n^2}$, we find

$$\frac{k}{2n^2+k} < \sqrt{1+\frac{k}{n^2}} - 1 < \frac{k}{2n^2}.$$

Hence

$$\sum_{k=1}^{n} \frac{k}{2n^2+k} < s_n < \frac{1}{2n^2} \sum_{k=1}^{n} k.$$

The right member is equal to

$$\frac{1}{2n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{4n^2}.$$

Therefore the limit of the right member is equal to $\frac{1}{4}$ as $n \to \infty$. On the other hand,

$$\lim_{n\to\infty} \left(\frac{1}{2n^2} \sum_{k=1}^{n} k - \sum_{k=1}^{n} \frac{k}{2n^2+k} \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{2n^2(2n^2+k)}.$$

But

$$\sum_{k=1}^{n} \frac{k^2}{2n^2(2n^2+k)} < \sum_{k=1}^{n} \frac{k^2}{4n^4} = \frac{1^2+2^2+\ldots+n^2}{4n^4}.$$

Consequently

$$\lim_{n\to\infty} \left\{ \frac{1}{2n^2} \sum_{k=1}^{n} k - \sum_{k=1}^{n} \frac{k}{2n^2+k} \right\} = 0$$

$$\text{and} \quad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{2n^2+k} = \frac{1}{4}.$$

Thus, both variables, between which S_n is contained, tend to $\frac{1}{4}$. Therefore

$$\lim_{n\to\infty} S_n = \frac{1}{4}.$$

12. We have

$$x_n^2 = a + x_{n-1}.$$

It is easy to see that the variable x_n increases. Let us show that all its values remain less than some constant number. We have

$$x_{n-1}^2 - x_{n-1} - a < 0,$$

since $x_{n-1} < x_n$.

Hence

$$\left(x_{n-1} - \frac{\sqrt{4a+1}+1}{2}\right)\left(x_{n-1} + \frac{\sqrt{4a+1}-1}{2}\right) < 0.$$

But since the second bracketed expression exceeds zero, it must be $x_{n-1} < \frac{\sqrt{4a+1}+1}{2}$, i.e. the increasing variable x_{n-1} is bounded, and consequently has a limit. Put $\lim\limits_{n\to\infty} x_{n-1} = \lim\limits_{n\to\infty} x_n = \alpha$. From the original relation between x_n and x_{n-1} we get

$$\alpha^2 - \alpha - a = 0,$$

and since $\alpha \geqslant 0$, we have

$$\alpha = \frac{\sqrt{4a+1}+1}{2}.$$

13. Let us prove that x_n is a decreasing variable. We have

$$x_{n+1} - x_n = \frac{1}{\sqrt{n+1}} - 2\left(\sqrt{n+1} - \sqrt{n}\right).$$

But

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{2\sqrt{n+1}}$$

and consequently

$$x_{n+1} < x_n.$$

But it is possible to prove (see Problem 6, Sec. 8) that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

Therefore

$$x_n > 2\left(\sqrt{n+1} - \sqrt{n}\right) - 2 > -2.$$

Thus, the decreasing variable x_n remains constantly greater than -2, hence, it has a limit.

14. Let us first show that $x_n > y_n$. Indeed

$$x_n - y_n = \frac{x_{n-1}+y_{n-1}}{2} - \sqrt{x_{n-1}y_{n-1}} = \frac{1}{2}(\sqrt{x_{n-1}} - \sqrt{y_{n-1}})^2 > 0.$$

But

$$x_n - x_{n-1} = \frac{x_{n-1}+y_{n-1}}{2} - x_{n-1} = \frac{y_{n-1}-x_{n-1}}{2} < 0;$$

$$x_{n-1} > x_n,$$

i.e. the variable x_n is a decreasing one. On the other hand,

$$y_n - y_{n-1} = \sqrt{y_{n-1}\cdot x_{n-1}} - y_{n-1} = \sqrt{y_{n-1}}(\sqrt{x_{n-1}} - \sqrt{y_{n-1}}) > 0,$$

i.e. $y_n > y_{n-1}$ and y_n is an increasing variable, wherefrom follows that each of the variables x_n and y_n has a limit. Put $\lim x_n = x$, $\lim y_n = y$. We have

$$x_n = \frac{x_{n-1}+y_{n-1}}{2}.$$

Hence

$$x = \frac{x+y}{2}$$

and consequently

$$x = y.$$

15. We have $\frac{1}{1-q} = s_1$, $\frac{1}{1-Q} = s$, hence $q = 1 - \frac{1}{s_1}$, $Q = 1 - \frac{1}{s}$. But

$$1 + qQ + q^2Q^2 + \ldots = \frac{1}{1-qQ} =$$

$$= \frac{1}{1-\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s}\right)} = \frac{ss_1}{s+s_1-1}.$$

16. We have

$$s = u_1 + u_1q + u_1q^2 + \ldots = u_1(1 + q + q^2 + \ldots),$$

$$\sigma^2 = u_1^2(1 + q^2 + q^4 + \ldots).$$

Further

$$s_n = \frac{u_nq - u_1}{q-1} = u_1\frac{1-q^n}{1-q} = s\cdot(1-q^n),$$

$$\sigma^2 = \frac{u_1^2}{1-q^2}, \quad s^2 = \frac{u_1^2}{(1-q)^2}.$$

We have

$$s^2+\sigma^2=\frac{2u_1^2}{(1-q)^2(1+q)}, \quad s^2-\sigma^2=\frac{2u_1^2q}{(1-q)^2(1+q)}.$$

Hence

$$q=\frac{s^2-\sigma^2}{s^2+\sigma^2}$$

and

$$s_n=s(1-q^n)=s\left\{1-\left[\frac{s^2-\sigma^2}{s^2+\sigma^2}\right]^n\right\}.$$

17. 1° Put $x=\frac{1}{y}$. Then $|y|>1$, and we may put $|y|=1+\rho$, where $\rho>0$.

We have

$$|n^kx^n|=\frac{n^k}{(1+\rho)^n}=$$

$$=\frac{n^k}{1+n\rho+\frac{n(n-1)}{1\cdot 2}\cdot\rho^2+\dots+\frac{n(n-1)\dots(n-k)}{1\cdot 2\cdot 3\dots(k+1)}\cdot\rho^{k+1}+\dots+\rho^n}.$$

Assuming that $n>k$, we find

$$|n^kx^n|=\frac{n^k}{(1+\rho)^n}<\frac{n^k(k+1)!}{n(n-1)(n-2)\dots(n-k+1)(n-k)\rho^{k+1}}=$$

$$=\frac{(k+1)!}{\rho^{k+1}}\,\frac{1}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)(n-k)}.$$

But the expression

$$\frac{(k+1)!}{\rho^{k+1}}\,\frac{1}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)(n-k)}\to 0$$

if $n\to\infty$ (k constant).

Therefore, indeed

$$\lim n^kx^n=0 \text{ if } n\to\infty.$$

2° Put $\sqrt[n]{n}-1=\alpha$ $(\alpha>0)$. We then have $n=(1+\alpha)^n$ Hence

$$n=1+n\alpha+\frac{n(n-1)}{1\cdot 2}\alpha^2+\dots+\alpha^n.$$

Consequently

$$n > \frac{n(n-1)}{1\cdot 2}\alpha^2, \quad \alpha^2 < \frac{2}{n-1} < \frac{4}{n} \quad (n > 2).$$

And so

$$\alpha < \frac{2}{\sqrt{n}} \quad \text{and} \quad 0 < \sqrt[n]{n} - 1 < \frac{2}{\sqrt{n}} \quad (n > 2).$$

Now it is obvious that

$$\lim \sqrt[n]{n} = 1.$$

18. We have

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1},$$

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$$

(see Problem 40, Sec. 7).

But

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots =$$

$$= \lim_{n\to\infty}\left\{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}\right\} = \lim_{n\to\infty}\left\{1 - \frac{1}{n+1}\right\} = 1.$$

Thus

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots .$$

Analogously

$$\frac{1}{4} = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} + \ldots .$$

We can prove a more general formula

$$\frac{1}{1\cdot 2\cdot 3\ldots(q+1)} + \frac{1}{2\cdot 3\cdot 4\ldots(q+2)} + \ldots + \frac{1}{n(n+1)\ldots(q+n)} + \ldots = \frac{1}{q\cdot q!}$$

(see Problem 26, Sec. 9).

19. Suppose the series is a convergent one, i.e. suppose $S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ has a limit which is equal to S as $n \to \infty$.

Then $\lim\limits_{n\to\infty} S_{2n}=S$. But on the other hand,

$$S_{2n}-S_n=\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}>\frac{1}{2}$$

(see Problem 1, Sec. 8) which is impossible. Thus, the series cannot be a convergent one. However, the divergence of this series can be proved in a different way. Let $2^k<n<$ $<2^{k+1}$. We then have

$$S_n=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots+$$
$$+\left(\frac{1}{2^{k-1}+1}+\dots+\frac{1}{2^k}\right)+\frac{1}{2^k+1}+\dots+\frac{1}{n}.$$

But

$$\frac{1}{3}+\frac{1}{4}>\frac{2}{4}=\frac{1}{2},\quad \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{4}{8}=\frac{1}{2},\ \dots .$$

Therefore

$$S_n>1+\frac{k}{2}.$$

But as $n\to\infty$, also $k\to\infty$, and consequently $S_n\to\infty$, hence, the series is a divergent one (see also Problem 22).

20. Put $S_n=1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\dots+\frac{1}{n^\alpha}$. To prove that the series is a convergent one it is necessary to prove that $\lim\limits_{n\to\infty} S_n$ exists. But it is easily seen that S_n increases along with an increase in n. It remains to prove that S_n is bounded. Let $2^{k-1}<n\leqslant 2^k$. We have

$$S_n\leqslant 1+\left(\frac{1}{2^\alpha}+\frac{1}{3^\alpha}\right)+\left(\frac{1}{4^\alpha}+\frac{1}{5^\alpha}+\frac{1}{6^\alpha}+\frac{1}{7^\alpha}\right)+\dots+$$
$$+\left(\frac{1}{(2^{k-1})^\alpha}+\frac{1}{(2^{k-1}+1)^\alpha}+\dots+\frac{1}{(2^k-1)^\alpha}\right).$$

But

$$\frac{1}{2^\alpha}+\frac{1}{3^\alpha}<2\,\frac{1}{2^\alpha}=\frac{1}{2^{\alpha-1}},$$
$$\frac{1}{4^\alpha}+\frac{1}{5^\alpha}+\frac{1}{6^\alpha}+\frac{1}{7^\alpha}<\frac{4}{4^\alpha}=\frac{1}{4^{\alpha-1}},$$
$$\dots\dots\dots\dots\dots\dots$$
$$\frac{1}{(2^{k-1})^\alpha}+\frac{1}{(2^{k-1}+1)^\alpha}+\dots+\frac{1}{(2^k-1)^\alpha}<\frac{2^{k-1}}{(2^{k-1})^\alpha}=\frac{1}{(2^{k-1})^{\alpha-1}}.$$

And so

$$S_n \leqslant 1+\frac{1}{2^{\alpha-1}}+\frac{1}{(2^2)^{\alpha-1}}+\ldots+\frac{1}{(2^{k-1})^{\alpha-1}}$$

or

$$S_n \leqslant 1+\frac{1}{2^{\alpha-1}}+\frac{1}{(2^2)^{\alpha-1}}+\ldots+\frac{1}{(2^{k-1})^{\alpha-1}}+\ldots,$$

$$S_n \leqslant \frac{1}{1-\frac{1}{2^{\alpha-1}}}.$$

Thus, S_n is really bounded, $\lim\limits_{n\to\infty} S_n$ exists and the series converges.

21. 1° We have (see Problem 22, Sec. 7)

$$1x+2x^2+\ldots+nx^n=\frac{x}{(x-1)^2}\{nx^{n+1}-(n+1)x^n+1\},$$

$$1+2x+3x^2+\ldots+nx^{n-1}+\ldots=$$
$$=\lim_{n\to\infty}\{1+2x+3x^2+\ldots+nx^{n-1}\}=$$
$$=\lim_{n\to\infty}\frac{1}{(x-1)^2}\{nx^{n+1}-(n+1)x^n+1\}=\frac{1}{(x-1)^2},$$

since

$$\lim_{n\to\infty} nx^n=0 \quad (|x|<1)$$

(see Problem 17, 1°).

2°, 3° From the results of Problem 33, Sec. 7 we get

$$1+4x+9x^2+\ldots+n^2x^{n-1}+\ldots=\frac{1+x}{(1-x)^3},$$

$$1+2^3x+3^3x^2+\ldots+n^3x^{n-1}+\ldots=\frac{1+4x+x^2}{(1-x)^4}.$$

22. 1° Follows immediately from Problem 41, Sec. 8. Hence, we can obtain one more proof of divergence of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots.$$

Put

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

Since the variable $\left(1+\frac{1}{n}\right)^n$ tends to e in an increasing manner, we have

$$\left(1+\frac{1}{n}\right)^n < e$$

for any whole positive n.

Hence

$$n \log\left(1+\frac{1}{n}\right) < 1$$

if the logarithm is taken to the base e. Or

$$\frac{1}{n} > \log\left(1+\frac{1}{n}\right),$$

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} > \log 2+\log\left(1+\frac{1}{2}\right)+$$
$$+\log\left(1+\frac{1}{3}\right)+\ldots+\log\left(1+\frac{1}{n}\right)=$$
$$=\log\frac{2\cdot 3\cdot 4\ldots(n+1)}{1\cdot 2\cdot 3\ldots n}=\log(n+1).$$

Hence

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} > \log(n+1)$$

and we get a divergent series.

2° Using the binomial formula, we obtain

$$\left(1+\frac{1}{n}\right)^n = 1+n\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\frac{1}{n^2}+$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3}+\ldots+$$
$$+\frac{n(n-1)(n-2)\ldots[n-(n-1)]}{1\cdot 2\cdot 3\ldots n}\cdot\frac{1}{n^n}=$$
$$=2+\frac{1}{1\cdot 2}\left(1-\frac{1}{n}\right)+\frac{1}{1\cdot 2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+$$
$$+\frac{1}{1\cdot 2\cdot 3\ldots n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right).$$

Put for brevity

$$\frac{1}{1\cdot 2\cdot 3\ldots k}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)=u_k.$$

Then

$$\left(1+\frac{1}{n}\right)^n = 2+u_2+u_3+\ldots+u_k+u_{k+1}+u_{k+2}+\ldots+u_n.$$

We have

$$u_k < \frac{1}{1\cdot2\cdot3\ldots k}, \qquad \frac{u_{k+1}}{u_k} = \frac{1-\frac{k}{n}}{k+1} < \frac{1}{k+1}.$$

Hence

$$u_{k+1} < u_k \frac{1}{k+1},$$

$$u_{k+2} < u_{k+1}\frac{1}{k+2} < u_k \frac{1}{(k+1)^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n < u_k \frac{1}{(k+1)^{n-k}}.$$

And so

$$u_{k+1}+u_{k+2}+\ldots+u_n <$$

$$< \frac{u_k}{k+1}\left[1+\frac{1}{k+1}+\ldots+\frac{1}{(k+1)^{n-k-1}}\right] < \frac{u_k}{k}.$$

Consequently

$$u_{k+1}+u_{k+2}+\ldots+u_n < \frac{1}{1\cdot2\cdot3\ldots k}\cdot\frac{1}{k}.$$

Hence

$$0 < \left(1+\frac{1}{n}\right)^n - (2+u_2+\ldots+u_k) < \frac{1}{1\cdot2\ldots k}\cdot\frac{1}{k}.$$

Let $n\to\infty$. Then

$$\lim_{n\to\infty} u_k = \frac{1}{1\cdot2\ldots k}$$

and, consequently,

$$0 < e - \left(2+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\ldots+\frac{1}{1\cdot2\cdot3\ldots k}\right) < \frac{1}{1\cdot2\ldots k}\cdot\frac{1}{k}.$$

wherefrom follows

$$e = 2+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\ldots+\frac{1}{1\cdot2\cdot3\ldots k}+\frac{\theta}{1\cdot2\cdot3\ldots k\cdot k}$$

$$(0<\theta<1).$$

Thus, we may write

$$e=2+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots+\frac{1}{1\cdot 2\cdot 3\ldots k}+\ldots .$$

23. We have

$$2\sin\frac{1}{2}x-\sin x=2\sin\frac{1}{2}x\left(1-\cos\frac{1}{2}x\right)=4\sin\frac{1}{2}x\sin^2\frac{1}{4}x.$$

Hence

$$2\sin\frac{1}{2}x-\sin x<4\frac{x}{2}\left(\frac{x}{4}\right)^2,$$

since $\sin\alpha<\alpha$ for $\alpha>0$.
Differently

$$2\sin\frac{1}{2}x-\sin x<\frac{1}{8}x^3. \qquad (1)$$

Replacing here x by $\frac{1}{2}x$, $\frac{1}{4}x$, $\ldots$, $\frac{1}{2^{n-1}}x$, we find

$$2\sin\frac{1}{4}x-\sin\frac{1}{2}x<\frac{1}{8}\left(\frac{x}{2}\right)^3, \qquad (2)$$

$$2\sin\frac{1}{8}x-\sin\frac{1}{4}x<\frac{1}{8}\left(\frac{x}{4}\right)^3, \qquad (3)$$

. . . .

$$2\sin\frac{1}{2^n}x-\sin\frac{1}{2^{n-1}}x<\frac{1}{8}\left(\frac{x}{2^{n-1}}\right)^3. \qquad (n)$$

Multiplying inequalities (1), (2), ..., (n) successively by 1, 2, ..., 2^{n-1} and adding them, we get

$$2^n\sin\frac{1}{2^n}x-\sin x<\frac{1}{8}x^3\left\{1+\frac{1}{2^2}+\frac{1}{4^2}+\ldots+\frac{1}{2^{2n-2}}\right\}.$$

Passing to the limit as $n\to\infty$, we find

$$\lim\left\{\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}x-\sin x\right\}\leqslant$$

$$\leqslant\frac{1}{8}x^3\lim\left\{1+\frac{1}{4^1}+\frac{1}{4^2}+\ldots+\frac{1}{4^{n-1}}\right\}.$$

But

$$\lim \left\{1+\frac{1}{4}+\frac{1}{4^2}+\ldots+\frac{1}{4^{n-1}}\right\}=\frac{1}{1-\frac{1}{4}}=\frac{4}{3},$$

$$\lim_{n\to\infty} \frac{\sin \frac{x}{2^n}}{\frac{x}{2^n}}=1.$$

Consequently

$$x-\sin x \leqslant \frac{1}{6} x^3.$$

24. 1° Put

$$S_n=\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n}.$$

It is required to prove that S_n has a limit as $n \to \infty$. As is easily seen, S_n increases along with an increase in n so that $S_{n+1} \geqslant S_n$. Let us prove that S_n is bounded. We have

$$S_n=\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n} \leqslant 9\left(\frac{1}{10}+\frac{1}{10^2}+\ldots+\frac{1}{10^n}\right)<$$
$$<9\left(\frac{1}{10}+\frac{1}{10^2}+\ldots+\frac{1}{10^n}+\ldots\right).$$

And so, $S_n<1$ and the series converges.

2° Since ω lies in the interval between 0 and 1, let us divide this interval into ten equal parts. In this event the number ω will be found either inside one of the subintervals or at its boundary. Consequently, we can find a whole number a_1 $(0 \leqslant a_1 \leqslant 9)$, such that

$$\frac{a_1}{10} \leqslant \omega < \frac{a_1+1}{10},$$

i.e.

$$0 \leqslant \omega-\frac{a_1}{10}<\frac{1}{10}.$$

Thus, the number $\omega-\frac{a_1}{10}$ lies in the interval between 0 and $\frac{1}{10}$. Let us divide this interval into ten equal parts.

Then we shall have

$$\frac{a_2}{10^2} \leqslant \omega - \frac{a_1}{10} < \frac{a_2+1}{10^2}.$$

Hence

$$\frac{a_1}{10} + \frac{a_2}{10^2} \leqslant \omega < \frac{a_1}{10} + \frac{a_2+1}{10^2}.$$

This operation can be continued in a similar way. Let us prove that

$$\lim_{n\to\infty}\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right) = \omega.$$

Here the variable increases but remains all the time less than $\frac{a_1+1}{10}$, consequently, it has a limit. Consider the variable

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{a_n+1}{10^n}.$$

It is easily seen that this variable decreases but remains greater than $\frac{a_1}{10}$ and, consequently, also has a limit. Since the difference

$$\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{a_n+1}{10^n}\right) - \left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right) = \frac{1}{10^n}$$

tends to zero as $n \to \infty$, both of these variables tend to one and the same limit, which, by virtue of the inequalities

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} \leqslant \omega < \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_{n-1}}{10^{n-1}} + \frac{a_n+1}{10^n},$$

will be equal to ω.

3° If the fraction is finite, then, there is no doubt, it is equal to a rational number. Let us pass over to the case of periodicity. In this case we have

$$\omega = \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} + $$
$$+ \frac{1}{10^n}\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right) + $$
$$+ \frac{1}{10^{2n}}\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right) + \ldots =$$
$$= \left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right)\left(1 + \frac{1}{10^n} + \frac{1}{10^{2n}} + \ldots\right) =$$
$$= \left(\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}\right)\frac{1}{1 - \frac{1}{10^n}} =$$
$$= \frac{a_1 10^{n-1} + a_2 10^{n-2} + \ldots + a_{n-1}10 + a_n}{10^n - 1},$$

i.e. ω is a rational number.

Likewise we make sure that a mixed periodic fraction (i.e. such a fraction whose period begins not with a_1, but later) will also be rational.

Making use of some arithmetic reasons, we can prove the converse; namely, if a number is rational, then its expansion into a decimal fraction will necessarily be either finite, or periodic (purely periodic, or mixed periodic).

Thus, every non-periodic infinite fraction necessarily yields an irrational number.

25. Suppose ω is rational, i.e. $\omega = \frac{Z}{N}$, where Z and N are whole numbers.

We have

$$\frac{Z}{N} = \frac{1}{l} + \frac{1}{l^4} + \frac{1}{l^9} + \ldots + \frac{1}{l^{n^2}} + \frac{1}{l^{(n+1)^2}} + \frac{1}{l^{(n+2)^2}} + \ldots .$$

Let us multiply both members of the equality by $l^{n^2}N$ and transpose the first n terms from the right to the left.

We get

$$Zl^{n^2} - N\left(l^{n^2-1} + l^{n^2-4} + \ldots + l^{n^2-(n-1)^2} + 1\right) =$$
$$= N\left\{\frac{1}{l^{2n+1}} + \frac{1}{l^{4n+4}} + \frac{1}{l^{6n+9}} + \ldots\right\}.$$

Hence

$$|Zl^{n^2}-N(l^{n^2-1}+l^{n^2-4}+\ldots+1)|<$$

$$<N\left\{\frac{1}{l^{2n+1}}+\frac{1}{l^{2(2n+1)}}+\frac{1}{l^{3(2n+1)}}+\ldots\right\}=N\frac{\frac{1}{l^{2n+1}}}{1-\frac{1}{l^{2n+1}}}.$$

And so

$$|Zl^{n^2}-N(l^{n^2-1}+l^{n^2-4}+\ldots+1)|<N\frac{1}{l^{2n+1}-1}.$$

If n is taken sufficiently large, then the right member can be made infinitely small, whereas the left member is an integer not equal to zero.

2° Proved as 1°.

26. We have

$$e=2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\frac{1}{(n+1)!}+\ldots.$$

Put

$$e=\frac{Z}{N}$$

(where Z and N are positive integers).

Then

$$\frac{Z}{N}=2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{N!}+\frac{1}{(N+1)!}+\ldots$$

or

$$Z(N-1)!-\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{N!}\right)N!=$$

$$=\frac{1}{N+1}+\frac{1}{(N+1)(N+2)}+\ldots.$$

Hence

$$\left|Z(N-1)!-\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{N!}\right)N!\right|<$$

$$<\frac{1}{N+1}+\frac{1}{(N+1)^2}+\frac{1}{(N+1)^3}+\ldots=\frac{1}{N},$$

which is impossible, since on the right we have a regular fraction, and on the left a whole number not equal to zero. Thus, e is an irrational number. If e is represented as a decimal fraction, then it will be an infinite non-periodic fraction. Given below is a value of e accurate to 2 500 decimal places.

e=2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995
95749 66967 62772 40766 30353 54759 45713 82178 52516 64274
27466 39193 20030 59921 81741 35966 29043 57290 03342 95260
59563 07381 32328 62794 34907 63233 82988 07531 95251 01901
15738 34187 93070 21540 89149 93488 41675 09244 76146 06680
82264 80016 84774 11853 74234 54424 37107 53907 77449 92069
55170 27618 38606 26133 13845 83000 75204 49338 26560 29760
67371 13200 70932 87091 27443 74704 72306 96977 20931 01416
92836 81902 55151 08657 46377 21112 52389 78442 50569 53696
77078 54499 69967 94686 44549 05987 93163 68892 30098 79312
77361 78215 42499 92295 76351 48220 82698 95193 66803 31825
28869 39849 64651 05820 93923 98294 88793 32036 25094 43117
30123 81970 68416 14039 70198 37679 32068 32823 76464 80429
53118 02328 78250 98194 55815 30175 67173 61332 06981 12509
96181 88159 30416 90351 59888 85193 45807 27386 67385 89422
87922 84998 92086 80582 57492 79610 48419 84443 63463 24496
84875 60233 62482 70419 78623 20900 21609 90235 30436 99418
49146 31409 34317 38143 64054 62531 52096 18369 08887 07016
76839 64243 78140 59271 45635 49061 30310 72085 10383 75051
01157 47704 17189 86106 87396 96552 12671 54688 95703 50354
02123 40784 98193 34321 06817 01210 15627 88023 51930 33224
74501 58539 04730 41995 77770 93503 66041 69973 29725 08868
76966 40355 57071 62268 44716 25607 98826 51787 13419 51246
65201 03059 21236 67719 43252 78675 39855 89448 96970 96409
75459 18569 56380 23637 01621 12047 74272 28364 89613 42251
64450 78182 44235 29486 36372 14174 02388 93441 24796 35743
70263 75529 44483 37998 01612 54922 78509 25778 25620 92622
64832 62779 33386 56648 16277 25164 01910 59004 91644 99828
93150 56604 72580 27786 31864 15519 56532 44258 69829 46959
30801 91529 87211 72556 34754 63964 47910 14590 40905 86298
49679 12874 06870 50489 58586 71747 98546 67757 57320 56812
88459 20541 33405 39220 00113 78630 09455 60688 16674 00169
84205 58040 33637 95376 45203 04024 32256 61352 78369 51177
88386 38744 39662 53224 98506 54995 88623 42818 99707 73327
61717 83928 03494 65014 34558 89707 19425 86398 77275 47109
62953 74152 11151 36835 06275 26023 26484 72870 39207 64310
05958 41166 12054 52970 30236 47254 92966 69381 15137 32275
36450 98889 03136 02057 24817 65851 18063 03644 28123 14965
50704 75102 54465 01172 72115 55194 86685 08003 68532 28183
15219 60037 35625 27944 95158 28418 82947 87610 85263 98139
55990 06737 64829 22443 75287 18462 45780 36192 98197 13991
47564 48826 26039 03381 44182 32625 15097 48279 87779 96437
30899 70388 86778 22713 83605 77297 88241 25611 90717 66394
65070 63304 52795 46618 55096 66618 56647 09711 34447 40160
70462 62156 80717 48187 78443 71436 98821 85596 70959 10259
68620 02353 71858 87485 69652 20005 03117 34392 07321 13908
03293 63447 97273 55955 27734 90717 83793 42163 70120 50054
51326 38354 40001 86323 99149 07054 79778 05669 78533 58048
96690 62951 19432 47309 95876 55236 81285 90413 83241 16072
26029 98330 53537 08761 38939 63917 79574 54016 13722 36133

Let us also give the logarithm of this number to base 10 accurate to 282 decimal places.

$\log_{10} e$=0.43429 44819 03251 82765 11289
18916 60508 22943 97005 80366
65661 14453 78316 58646 49208
87077 47292 24949 33843 17483
18706 10674 47663 03733 64167
92871 58963 90656 92210 64662
81226 58521 27086 56867 03295
93370 86965 88266 88331 16360
77384 90514 28443 48666 76864
65860 85135 56148 21234 87653
43543 43573 17247 48049 05993
55353 05

27. It is easily seen, that if l_k (beginning with some k) are all equal to one another, then we deal with an infinitely decreasing geometric progression, and ω is rational indeed. It remains to prove that if such circumstance (equality of all l_k beginning with some k) does not take place, then ω is irrational. It can be proved in the same way as in Problem 25.

28. Let us prove that the variable u_n decreases, i.e. that $u_{n+1} < u_n$. We have

$$u_{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\frac{1}{n+1}-\log(n+1).$$

Hence

$$u_{n+1}-u_n=\frac{1}{n+1}-\log(n+1)+\log n=\frac{1}{n+1}-\log\left(1+\frac{1}{n}\right).$$

Consider the variable

$$v_n=\left(1+\frac{1}{n}\right)^{n+1}$$

and prove that it decreases, i.e. that $v_{n+1}<v_n$ or that

$$\left(1+\frac{1}{n+1}\right)^{n+2}<\left(1+\frac{1}{n}\right)^{n+1},$$

i.e. show that

$$\left(1+\frac{1}{n}\right)^{\frac{n+1}{n+2}}>1+\frac{1}{n+1}.$$

We have $(1+\alpha)^{\frac{m}{n}}>1+\alpha\frac{m}{n}$ (see Problem 40, 1°, Sec. 8).

Replacing here α by $\frac{1}{n}$, and $\frac{m}{n}$ by $\frac{n+1}{n+2}$, we find

$$\left(1+\frac{1}{n}\right)^{\frac{n+1}{n+2}} > 1+\frac{1}{n}\frac{(n+1)}{(n+2)}.$$

But

$$1+\frac{n+1}{n(n+2)} > 1+\frac{1}{n+1}.$$

And so, the variable $v_n=\left(1+\frac{1}{n}\right)^{n+1}$ decreases. Let us show that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e.$$

We have

$$\left(1+\frac{1}{n}\right)^n=\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)}.$$

But $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)=1$. Thus, indeed $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e$ and consequently

$$\left(1+\frac{1}{n}\right)^{n+1} > e.$$

Therefore $(n+1)\log\left(1+\frac{1}{n}\right) > 1$, $\log\left(1+\frac{1}{n}\right) > \frac{1}{n+1}$, and

$$u_{n+1}-u_n < 0,$$

and the variable u_n is a decreasing one.

On the other hand,

$$u_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n > \log(n+1)-$$
$$-\log n > \log\left(1+\frac{1}{n}\right) > 0.$$

Since the variable u_n decreases but remains greater than zero, it has a limit. Let us denote this limit by C.

$$C=\lim\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right\}.$$

C is called Euler's constant. Let us give the value of this constant accurate to 263 decimal places.

$$\begin{aligned} C = 0.57721\ &56649\ 01532\ 86060\ 65120\\ 90082\ &40243\ 10421\ 59335\ 93992\\ 35988\ &05767\ 23488\ 48677\ 26777\\ 66467\ &09369\ 47063\ 29174\ 67495\\ 14631\ &44724\ 98070\ 82480\ 96050\\ 40144\ &86542\ 83622\ 41739\ 97644\\ 92353\ &62535\ 00333\ 74293\ 73377\\ 37673\ &94279\ 25952\ 58247\ 09491\\ 60087\ &35203\ 94816\ 56708\ 53233\\ 15177\ &66115\ 28621\ 19950\ 15079\\ 84793\ &74508\ 569 \end{aligned}$$

29. We have

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2},$$

$$\sin\frac{x}{2} = 2\sin\frac{x}{2^2}\cos\frac{x}{2^2},$$

$$\sin\frac{x}{2^2} = 2\sin\frac{x}{2^3}\cos\frac{x}{2^3},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\sin\frac{x}{2^{n-1}} = 2\sin\frac{x}{2^n}\cos\frac{x}{2^n}.$$

Multiplying these equalities, we find

$$\sin x = 2^n \sin\frac{x}{2^n}\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n}.$$

Then

$$\frac{2^n\sin\frac{x}{2^n}}{\sin x} = \frac{1}{\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n}}.$$

We have

$$\lim_{n\to\infty} 2^n \sin\frac{x}{2^n} = \lim \frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\,x = x.$$

Put

$$\lim_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n}\right) =$$

$$= \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots.$$

Then we have

$$\frac{x}{\sin x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\dots .$$

Putting here $x = \frac{\pi}{2}$, we find the required formula. The number π, like e, is irrational and, consequently, cannot be expressed by a finite or periodic decimal fraction. Given below is the value of π accurate to 2035 decimal places.

π=3.14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
71010	00313	78387	52886	58753	32083	81420	61717	76691	47303
59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
55478	57242	45415	06959	50829	53311	68617	27855	88907	50983
81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
25338	24300	35587	64024	74694	73263	91419	92726	04269	92279
67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
16136	11573	52552	13347	57418	49648	43852	33239	07394	14333
45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
46776	46575	73962	41389	08658	32645	99581	33904	78027	59009
94657	64078	95126	94683	98352	59570	98258			